A RESULT CONCERNING DERIVATIONS IN BANACH ALGEBRAS

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ABSTRACT. The main result: Let A be a Banach algebra over the complex field C . Suppose there exists a continuous derivation $D: A \rightarrow A$, such that $\alpha D^3 + D^2$ is a derivation for some $\alpha \in C$. In this case D maps A into its radical.

This paper is a continuation of our earlier work [7, 8]. Let R be a ring with center $Z(R)$. We write $[x, y]$ for $xy - yx$ and use the identities $[xy, z] = [x, z]y + x[y, z]$, $[x, yz] = [x, y]z + y[x, z]$. An additive mapping $D: R \rightarrow R$ is called a derivation if $D(xy) = D(x)y + xD(y)$, $x, y \in R$. A derivation D is inner if there exists $a \in R$, such that $D(x) = [a, x]$ holds for all $x \in R$. Recall that a ring R is prime if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$.

The main purpose of this paper is to prove the following result.

Theorem 1. *Let A be a Banach algebra over the complex field C . Suppose there exists a continuous derivation $D: A \rightarrow A$, such that $\alpha D^3 + D^2$ is a derivation for some $\alpha \in C$. In this case D maps A into its radical.*

In the proof of Theorem 1 we need the result below, which might be of some independent interest. The proof of this purely algebraic result is rather long, but it is elementary in the sense that one can follow the proof without any specific knowledge concerning prime rings.

Theorem 2. *Let R be a prime ring of characteristic different from two. Suppose there exists derivations D_1, D_2 , and D_3 on R , such that $D_1(D_1^2(x) + D_2(x)) = D_3(x)$ holds for $x \in R$. In this case we have either $D_1 = 0$ or $D_2 = 0$.*

Proof. Let us introduce a mapping $F: R \rightarrow R$ by the relation $F(x) = D_1^2(x) + D_2(x)$. A routine calculation shows that

$$(1) \quad F(xy) = F(x)y + xF(y) + 2D_1(x)D_1(y)$$

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holds for all pairs $x, y \in R$. Now the assumption of the theorem can be written in the form

$$(2) \quad D_1(F(x)) = D_3(x), \quad x \in R.$$

Let x be xy in (2). Then according to (1), we have

$$\begin{aligned} 0 &= D_1(F(xy)) - D_3(xy) \\ &= D_1(F(x)y + xF(y) + 2D_1(x)D_1(y)) - D_3(x)y - xD_3(y) \\ &= D_1(F(x))y + F(x)D_1(y) + D_1(x)F(y) + xD_1(F(y)) \\ &\quad + 2D_1^2(x)D_1(y) + 2D_1(x)D_1^2(y) - D_3(x)y - xD_3(y). \end{aligned}$$

According to (2), the above calculation reduces to

$$F(x)D_1(y) + D_1(x)F(y) + 2D_1^2(x)D_1(y) + 2D_1(x)D_1^2(y) = 0,$$

which can be written in the form

$$(3) \quad H(x)D_1(y) + D_1(x)H(y) = 0, \quad x, y \in R,$$

where $H(x)$ stands for $3D_1^2(x) + D_2(x)$. It is easy to see that

$$(4) \quad H(xy) = H(x)y + xH(y) + 6D_1(x)D_1(y)$$

holds for all pairs $x, y \in R$. Let y be yz in (3). Then according to (4), we have

$$\begin{aligned} 0 &= H(x)D_1(yz) + D_1(x)H(yz) \\ &= H(x)D_1(y)z + H(x)yD_1(z) + D_1(x)H(y)z \\ &\quad + D_1(x)yH(z) + 6D_1(x)D_1(y)D_1(z). \end{aligned}$$

Since (3) holds, the above calculation gives

$$(5) \quad H(x)yD_1(z) + D_1(x)yH(z) + 6D_1(x)D_1(y)D_1(z) = 0, \quad x, y, z \in R.$$

For $y = D_1(x)$ and $z = y$ the relation (5) gives

$$(6) \quad H(x)D_1(x)D_1(y) + D_1(x)^2H(y) + 6D_1(x)D_1^2(x)D_1(y) = 0, \quad x, y \in R.$$

Left multiplication of (3) by $D_1(x)$ gives

$$(7) \quad D_1(x)H(x)D_1(y) + D_1(x)^2H(y) = 0, \quad x, y \in R.$$

Combining (6) with (7) one obtains

$$(H(x)D_1(x) - D_1(x)H(x))D_1(y) + 6D_1(x)D_1^2(x)D_1(y) = 0.$$

In the above relation, one can replace $H(x)D_1(x) - D_1(x)H(x)$ by $-2D_1(x)H(x)$ because of (3). Thus, since R is not of characteristic 2, we have

$$D_1(x)(H(x) - 3D_1^2(x))D_1(y) = 0$$

and finally

$$(8) \quad D_1(x)D_2(x)D_1(y) = 0, \quad x, y \in R,$$

because $H(x) = 3D_1^2(x) + D_2(x)$. Putting zy instead of y in (8) we arrive at

$$D_1(x)D_2(x)zD_1(y) = 0, \quad x, y, z \in R.$$

From the above relation, one obtains that either $D_1 = 0$ or $D_1(x)D_2(x) = 0$ for all $x \in R$. In any case

$$(9) \quad D_1(x)D_2(x) = 0, \quad x \in R.$$

The linearization of (9) leads to

$$(10) \quad D_1(x)D_2(y) + D_1(y)D_2(x) = 0, \quad x, y \in R.$$

Putting yz instead of y in (10) we obtain using (10) again

$$\begin{aligned} 0 &= D_1(x)D_2(yz) + D_1(yz)D_2(x) \\ &= D_1(x)D_2(y)z + D_1(x)yD_2(z) + D_1(y)zD_2(x) + yD_1(z)D_2(x) \\ &= -D_1(y)D_2(x)z + D_1(x)yD_2(z) + D_1(y)zD_2(x) - yD_1(x)D_2(z). \end{aligned}$$

Thus we have

$$(11) \quad D_1(y)[z, D_2(x)] + [D_1(x), y]D_2(z) = 0,$$

and, in particular, for $z = D_2(x)$

$$(12) \quad [D_1(x), y]D_2^2(x) = 0, \quad x, y \in R.$$

Substituting yz for y in (12), we obtain

$$(13) \quad [D_1(x), y]zD_2^2(x) = 0, \quad x, y, z \in R.$$

Suppose that $D_2^2(a) \neq 0$ for some $a \in R$. In this case it follows from (13) that $[D_1(a), y] = 0$, $y \in R$, by primeness of R . In other words $D_1(a) \in Z(R)$. Now putting $x = a$ in (9) and multiplying (9) from the left by x , we obtain

$$D_1(a)xD_2(a) = 0, \quad x \in R,$$

whence it follows that $D_1(a) = 0$, otherwise $D_2(a)$ would be zero, which contradicts the assumption $D_2^2(a) \neq 0$. Now for $y = a$, the relation (10) reduces to $D_1(x)D_2(a) = 0$, $x \in R$, whence putting $x = a$ in (11) we obtain

$$D_1(x)yD_2(a) = 0, \quad x, y \in R,$$

which implies $D_1 = 0$, since $D_2(a) \neq 0$. Therefore we have proved that $D_1 = 0$ in case $D_2^2(a) \neq 0$ for some $a \in R$. In other words $D_2^2(x) = 0$ for all $x \in R$ if $D_1 \neq 0$. But in this case we have $D_2 = 0$ by Theorem 1 in [3]. The proof of the theorem is complete.

Theorem 2 is in the spirit of a classical result of Posner [3, Theorem 1], which states that if R is a prime ring of a characteristic different from two and D , G are nonzero derivations of R , then DG cannot be a derivation.

Proof of Theorem 1. Sinclair [4] has proved that any continuous linear derivation of a Banach algebra leaves the primitive ideals invariant. Since the radical of A is the intersection of all primitive ideals, any continuous linear derivation leaves the radical invariant as well, which means that there is no loss of generality in assuming that A is semisimple. Since D leaves all primitive ideals invariant, one can introduce for any primitive ideal $P \subset A$ a derivation $D_P: A/P \rightarrow A/P$, where A/P is the factor algebra, by $D_P(\hat{x}) = D(x)$, $\hat{x} = x + P$. Singer and Wermer [5] have proved that any continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. By the result of Johnson and Sinclair [2], any linear derivation on a semisimple Banach algebra is continuous.

Hence there is no nonzero linear derivation on a commutative Banach algebra. Therefore in case A/P is a commutative Banach algebra, one can conclude that $D_P = 0$, because A/P is semisimple. It remains to prove that $D_P = 0$; also in the case, A/P is noncommutative. The assumption of the theorem that $\alpha D^3 + D^2$ is a derivation gives that $\alpha D_P^3 + D_P^2$ is a derivation. Let us first assume that $\alpha = 0$. In this case we have D_P^2 is a derivation, and since A/P is prime, it follows from [2, Theorem 1] that $D_P = 0$. In case $\alpha \neq 0$ all the assumptions of Theorem 2 are fulfilled (note that D_P stands for D_1 and D_P/α for D_2). Thus we have $D_P = 0$ or $D_P/\alpha = 0$. In any case $D_P = 0$. In other words, $D(x)$ is in the intersection of all primitive ideals of A for any $x \in A$. Since the intersection of all primitive ideals is the radical, the proof of the theorem is complete.

In the proof of Theorem 1 we used the Singer-Wermer theorem, which states that any continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. Thomas [6] has generalized the Singer-Wermer theorem by proving that the continuity assumption is superfluous. This leads to the question of whether Theorem 1 can be proved without any continuity assumptions. We do not know whether the answer to this question is affirmative. However, in a special case when a Banach algebra is semisimple, one can prove the following result.

Theorem 3. *Let A be a semisimple Banach algebra over the complex field C . Suppose there exists a linear derivation $D: A \rightarrow A$, such that $\alpha D^3 + D^2$ is a derivation for some $\alpha \in C$. In this case $D = 0$*

Proof. The proof goes through in the same way as the proof of Theorem 1 with the only exception that at the beginning of the proof one has to use the fact that any linear derivation on a semisimple Banach algebra is continuous (see [4]).

It is interesting that in Theorem 3 the $\alpha = 0$ case is included; because it is known (see the proof of Lemma 1.1.9 in [1]) that if D and D^2 are derivations of a semiprime ring (in particular a C^* -algebra), then $D = 0$. Let us point out that one can prove nothing in case D and D^3 are derivations. Namely if one takes a , which is not in the center such that $a^2 = 0$, $D(x) = [a, x]$, then an easy calculation shows that $D^3(x) = 0$ for all x , but $D \neq 0$.

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