CHARACTERISTIC CONDITIONS FOR A c_0 -SEMIGROUP WITH CONTINUITY IN THE UNIFORM OPERATOR TOPOLOGY FOR $t > 0$ IN HILBERT SPACE

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ABSTRACT. In this paper, we obtain necessary and sufficient conditions for a c_0 -semigroup with continuity in the uniform operator topology for $t > 0$ in Hilbert space in terms of the spectral property of its infinitesimal generator. Then we get the characteristic theorem for the compact semigroup.

1. INTRODUCTION AND MAIN RESULTS

Let H be a Hilbert space and $\exp(tA)$ ($t > 0$) be strongly continuous at the origin semigroup of bounded linear operators in H , briefly called a c_0 -semigroup. A c_0 -semigroup $\exp(tA)$ is called continuous in the uniform operator topology for $t > 0$ if

$$\lim_{h \rightarrow 0} \|\exp(t+h)A - \exp(tA)\| = 0 \quad \text{for any } t > 0.$$

As shown in [1], this class of c_0 -semigroup properly contains the compact semigroup, the differentiable semigroup, and the analytic semigroup; hence it is very wide. As shown in [2], it satisfies the spectrum determined growth assumption, which is a very important property concerned with the exponential stability of linear dynamical systems, so it is very useful. But the characteristic conditions, i.e., the spectral property of its infinitesimal generator for such kinds of semigroups, has been an open problem for many years. This is because it is such a difficult problem that needs a new idea and a new method to treat it. In this paper, we have solved this problem in Hilbert space and obtained the following theorem.

Theorem 1 (Main result). *Let $\exp(tA)$ be a c_0 -semigroup in Hilbert space H and $\|\exp(tA)\| < Me^{\sigma t}$ ($t > 0$), where M, σ are two constant numbers, $M > 0$. Then $\exp(tA)$ is continuous in the uniform operator topology for $t > 0$ if and only if*

$$\lim_{|w| \rightarrow \infty} \|R(\tau_0 + iw, A)\| = 0 \quad \text{for some } \tau_0 \geq \max\{0, \sigma\},$$

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where $R(\tau_0 + iw; A) = (\tau_0 + iw - A)^{-1}$ and A is the infinitesimal generator of $\exp(tA)$.

Then, by [1, §2.3, Theorem 3.3], we get Theorem 2 directly:

Theorem 2 (Characteristic theorem for compact semigroup). *Let $\exp(tA)$ be a c_0 -semigroup in Hilbert space H and $\|\exp(tA)\| \leq Me^{\sigma t}$, where M, σ are two constant numbers, $M > 0$. Then $\exp(tA)$ is a compact semigroup (i.e., $\exp(tA)$ is a compact operator in H for any $t > 0$) if and only if*

(i) *there exist some number $\tau_0 > \max\{0, \sigma\}$ such that*

$$\lim_{|w| \rightarrow \infty} \|R(\tau_0 + iw; A)\| = 0;$$

(ii) *$R(\lambda; A)$ is a compact operator for every $\lambda \in \rho(A)$, where $\rho(A)$ is the resolvent set of A .*

2. PROOF OF THEOREM 1

Before proving Theorem 1, we must first prove three lemmas. Let $L_p(R)$ ($p \geq 1$) be the space of all complex-valued Lebesgue measurable functions $f(t)$ on $R = (-\infty, +\infty)$ such that

$$\int_{-\infty}^{+\infty} |f(t)|^p dt < \infty.$$

Lemma 1. *Let $\exp(tA)$ be a c_0 -semigroup in a Hilbert space H and $\|\exp(tA)\| \leq Me^{\sigma t}$, where M, σ are two constant numbers, $M > 0$. Then for any $x \in H$, $\tau > \sigma$, as function of $w \in R$, $\|(\tau + iw - A)^{-1}x\|$, $\|(\tau - iw - A^*)^{-1}x\| \in L_2(R)$. Proof. See Lemma 1 of [2].*

Lemma 2. *Let A be a linear bounded operator in H and*

$$\lim_{|w| \rightarrow \infty} \|R(\tau_0 + iw; A)\| = 0 \quad \text{for some } \tau_0 > 0.$$

Then there exists a monotone increasing function

$$\phi(x): [0, \infty) \rightarrow [0, \infty), \quad \lim_{x \rightarrow \infty} \phi(x) = \infty,$$

such that

$$\Sigma = \{\lambda = \tau + iw \mid |\tau - \tau_0| \leq \frac{1}{2}\phi(|w|)\} \subset \rho(A)$$

and

$$(2.1) \quad \|R(\tau + iw; A)\| \leq 2\|R(\tau_0 + iw; A)\|,$$

$$(2.2) \quad \|R(\tau + iw; A)x\| \leq 2\|R(\tau_0 + iw; A)x\|$$

for any $\lambda = \tau + iw \in \Sigma$ and $x \in H$.

Proof. For every $x \geq 0$, let

$$\psi(x) = \max_{|w| \geq x} \|R(\tau_0 + iw; A)\| \quad (-\infty < w < +\infty).$$

Then $\psi(x)$ is a monotone decreasing function on $[0, \infty)$ and $\lim_{x \rightarrow \infty} \psi(x) = 0$.

Now, we set $\phi(x) = 1/\psi(x)$ for every $x \geq 0$. Then for every $\lambda = \tau + iw \in \Sigma$, we have

$$(2.3) \quad R(\tau + iw; A) = \sum_{k=0}^{\infty} R(\tau_0 + iw; A)^{k+1} (\tau - \tau_0)^k.$$

Here the series on the right of (2.3) is converged in the uniform operator topology when $|\tau - \tau_0| \leq \frac{1}{2}\phi(|w|)$, so $\Sigma \subset \rho(A)$. Therefore

$$\|R(\tau + iw; A)\| \leq \sum_{k=0}^{\infty} \|R(\tau_0 + iw; A)\|^{k+1} |\tau - \tau_0|^k \leq 2\|R(\tau_0 + iw; A)\|$$

and

$$\begin{aligned} \|R(\tau + iw; A)x\| &\leq \sum_{k=0}^{\infty} \|R(\tau_0 + iw; A)^{k+1} x (\tau - \tau_0)^k\| \\ &\leq \|R(\tau_0 + iw; A)x\| \sum_{k=0}^{\infty} \|R(\tau_0 + iw; A)\|^k |\tau - \tau_0|^k \\ &\leq 2\|R(\tau_0 + iw; A)x\| \end{aligned}$$

for every $\lambda = \tau + iw \in \Sigma$, $x \in H$. $\square$

Let $\exp(tA)$ be a c_0 -semigroup generated by A in H and $\|\exp(tA)\| < Me^{\sigma t}$, where M, σ are two constant numbers, $M > 0$. Then for $\tau_0 > \max\{0, \sigma\}$, the straight line $\operatorname{Re} \lambda = \tau_0$ is in $\rho(A)$. Defining linear bounded operator $T_n(t)$ on H ,

$$(2.4) \quad T_n(t)x = \frac{1}{2\pi i} \int_{\tau_0 - in}^{\tau_0 + in} e^{\lambda t} R(\lambda; A)x d\lambda,$$

where $x \in H$, $t > 0$, $n = 1, 2, 3, \dots$.

Lemma 3. If $\lim_{|w| \rightarrow \infty} \|R(\tau_0 + iw; A)\| = 0$, then

$$(2.5) \quad \lim_{n \rightarrow \infty} \|\exp(tA) - T_n(t)\| = 0 \quad \text{for every } t > 0$$

and the convergence is uniform on any finite interval of t .

Proof. Let $[t_1, t_2]$ be any finite interval of t , where $t_2 > t_1 > 0$.

First of all, we shall prove

$$\exp(tA)x = \lim_{n \rightarrow \infty} T_n(t)x \quad \text{for any } x \in H, \quad t \in [t_1, t_2].$$

In fact, from (2.4), we have

$$T_n(t)x = \frac{1}{2\pi i} \left[\frac{e^{\lambda t}}{t} R(\lambda; A)x \Big|_{\tau_0 - in}^{\tau_0 + in} + \int_{\tau_0 - in}^{\tau_0 + in} \frac{1}{t} e^{\lambda t} R^2(\lambda; A)x d\lambda \right].$$

So for every $y \in H$, using the inner product of H , we get

$$\begin{aligned} |\langle T_n(t)x, y \rangle| &\leq \frac{1}{2\pi} \cdot \frac{1}{t_1} \cdot e^{\tau_0 t_2} [\|R(\tau_0 + in; A)\| + \|R(\tau_0 - in; A)\|] \cdot \|x\| \cdot \|y\| \\ &\quad + \frac{1}{2\pi} \cdot \frac{1}{t_1} \cdot e^{\tau_0 t_2} \cdot \int_{-n}^n \|R(\tau_0 + iw; A)x\| \cdot \|R(\tau_0 - iw, A^*)y\| d\omega. \end{aligned}$$

Noting $\lim_{|w| \rightarrow \infty} \|R(\tau_0 + iw; A)\| = 0$ and Lemma 1, we have $|\langle T_n(t)x, y \rangle| \leq M(x, y)$, where $M(x, y)$ only depends on x and y . By the resonance theorem, there exists a positive constant number M_1 such that

$$(2.6) \quad \|T_n(t)\| \leq M_1 \quad (t_1 \leq t \leq t_2, \quad n = 1, 2, 3, \dots).$$

As shown in [1, §1.7, Corollary 7.5] we have

$$\exp(tA)x = \lim_{n \rightarrow \infty} T_n(t)x \quad \text{for any } x \in D(A^2)$$

and the convergence is uniform on $t \in [t_1, t_2]$. Noting $D(A^2)$ is dense in H and (2.6), we have

$$\exp(tA)x = \lim_{n \rightarrow \infty} T_n(t)x \quad \text{for every } x \in H$$

and the convergence is uniform on $t \in [t_1, t_2]$; i.e.,

$$(2.7) \quad \exp(tA)x = \frac{1}{2\pi i} \int_{\tau_0 - iw}^{\tau_0 + iw} e^{\lambda t} R(\lambda; A)x \, d\lambda, \quad x \in H, \quad t \in [t_1, t_2].$$

For any $x, y \in H$, from (2.7) we have

$$(2.8) \quad \begin{aligned} \langle T(t) - T_n(t)x, y \rangle &= \frac{1}{2\pi i} \lim_{|w| \rightarrow \infty} \int_{\tau_0 - iw}^{\tau_0 + iw} e^{\lambda t} \langle R(\lambda; A)x, y \rangle \, d\lambda \\ &\quad - \frac{1}{2\pi i} \int_{\tau_0 - in}^{\tau_0 + in} e^{\lambda t} \langle R(\lambda; A)x, y \rangle \, d\lambda \\ &= I_1(n) + I_2(n), \end{aligned}$$

where

$$\begin{aligned} I_1(n) &= \frac{1}{2\pi i} \lim_{|w| \rightarrow \infty} \int_{\tau_0 + in}^{\tau_0 + iw} e^{\lambda t} \langle R(\lambda; A)x, y \rangle \, d\lambda, \\ I_2(n) &= \frac{1}{2\pi i} \lim_{|w| \rightarrow \infty} \int_{\tau_0 - iw}^{\tau_0 - in} e^{\lambda t} \langle R(\lambda; A)x, y \rangle \, d\lambda. \end{aligned}$$

FIGURE 1

Because $e^{\lambda t} \langle R(\lambda; A)x, y \rangle$ is analytic in $\rho(A)$ and closed curve $\widehat{ABCD} \subset \Sigma \subset \rho(A)$, by the Cauchy-Goursat integral theorem, we have

$$\oint_{\widehat{ABCD}} e^{\lambda t} \langle R(\lambda; A)x, y \rangle d\lambda = 0,$$

i.e.,

$$\begin{aligned} & \int_{\tau_0+in}^{\tau_0+iw} e^{\lambda t} \langle R(\lambda; A)x, y \rangle d\lambda \\ &= \int_{\tau_0+in}^{\tau_n+in} e^{\lambda t} \langle R(\lambda; A)x, y \rangle d\lambda + \int_{\tau_n+in}^{\tau_n+iw} e^{\lambda t} \langle R(\lambda; A)x, y \rangle d\lambda \\ &+ \int_{\tau_n+iw}^{\tau_0+iw} e^{\lambda t} \langle R(\lambda; A)x, y \rangle d\lambda \end{aligned}$$

where τ_n is the real part of the crossing point of $\text{Im } \lambda = n$ and $|\tau - \tau_0| = \frac{1}{2}\phi(|w|)$ and $\lim_{n \rightarrow \infty} \tau_n = -\infty$. (See Figure 1.)

Because $\lim_{|w| \rightarrow \infty} \|R(\tau_0 + iw; A)\| = 0$, we have

$$\begin{aligned} \lim_{|w| \rightarrow \infty} \int_{\tau_n+in}^{\tau_n+iw} e^{\lambda t} \langle R(\lambda; A)x, y \rangle d\lambda &= -\frac{1}{t} e^{(\tau_n+in)t} \langle R(\tau_n + in; A)x, y \rangle \\ &+ \lim_{|w| \rightarrow \infty} \int_{\tau_n+in}^{\tau_n+iw} \frac{1}{t} e^{\lambda t} \langle R^2(\lambda; A)x, y \rangle d\lambda \end{aligned}$$

and

$$\begin{aligned} \left| \lim_{|w| \rightarrow \infty} \int_{\tau_n+in}^{\tau_0+iw} e^{\lambda t} \langle R(\lambda; A)x, y \rangle d\lambda \right| &\leq \lim_{|w| \rightarrow \infty} \int_{\tau_n}^{\tau_0} e^{\tau t} \|R(\tau + iw; A)\| \|x\| \|y\| d\tau \\ &\leq 2\|x\| \|y\| \cdot \frac{1}{t} e^{\tau_0 t} \lim_{|w| \rightarrow \infty} \|R(\tau_0 + iw; A)\| = 0. \end{aligned}$$

Therefore

$$\begin{aligned} I_1(n) &= \frac{1}{2\pi i} \int_{\tau_0+in}^{\tau_n+in} e^{\lambda t} \langle R(\lambda; A)x, y \rangle d\lambda - \frac{1}{2\pi i} \frac{1}{t} e^{(\tau_n+in)t} \langle R(\tau_n + in; A)x, y \rangle \\ (2.9) \quad &+ \frac{1}{2\pi i} \lim_{|w| \rightarrow \infty} \int_{\tau_n+in}^{\tau_n+iw} \frac{1}{t} e^{\lambda t} \langle R^2(\lambda; A)x, y \rangle d\lambda. \end{aligned}$$

In the same way, we can get

$$\begin{aligned} I_2(n) &= \frac{1}{2\pi i} \int_{\tau_n-in}^{\tau_0-in} e^{\lambda t} \langle R(\lambda; A)x, y \rangle d\lambda + \frac{1}{2\pi i} \frac{1}{t} e^{(\tau_n-in)t} \langle R(\tau_n - in; A)x, y \rangle \\ (2.10) \quad &+ \frac{1}{2\pi i} \lim_{|w| \rightarrow \infty} \int_{\tau_n-iw}^{\tau_n-in} \frac{1}{t} e^{\lambda t} \langle R^2(\lambda; A)x, y \rangle d\lambda. \end{aligned}$$

Set

$$\begin{aligned} T_n^{(1)}(t)x &= \frac{1}{2\pi i} \left[\int_{\tau_0+in}^{\tau_n+in} e^{\lambda t} \cdot R(\lambda; A)x d\lambda + \int_{\tau_n-in}^{\tau_0-in} e^{\lambda t} \cdot R(\lambda; A)x d\lambda \right], \\ T_n^{(2)}(t)x &= \frac{1}{2\pi i} \frac{1}{t} e^{(\tau_n-in)t} R(\tau_n - in; A)x \\ &- \frac{1}{2\pi i} \frac{1}{t} e^{(\tau_n+in)t} R(\tau_n + in; A)x \quad \text{for any } x \in H. \end{aligned}$$

From (2.8), (2.9), and (2.10), we have

$$\begin{aligned} & \langle (\exp(tA) - T_n(t) - T_n^{(1)}(t) - T_n^{(2)}(t))x, y \rangle \\ &= \frac{1}{2\pi i} \lim_{|w| \rightarrow \infty} \int_{\tau_n + in}^{\tau_n + iw} \frac{1}{t} e^{\lambda t} \langle R^2(\lambda; A)x, y \rangle d\lambda \\ &+ \frac{1}{2\pi i} \lim_{|w| \rightarrow \infty} \int_{\tau_n - iw}^{\tau_n - in} \frac{1}{t} e^{\lambda t} \langle R^2(\lambda; A)x, y \rangle d\lambda, \end{aligned}$$

because

$$\begin{aligned} & \left| \int_{\tau_n + in}^{\tau_n + iw} \frac{1}{t} e^{\lambda t} \langle R^2(\lambda; A)x, y \rangle d\lambda \right| \\ & \leq \frac{1}{t} e^{\tau_n t} \left| \int_n^w \langle R(\tau_n + i\mu; A)x, R(\tau_n - i\mu; A^*)y \rangle d\mu \right| \\ & \leq \frac{4}{t} e^{\tau_n t} \int_n^w \|R(\tau_0 + i\mu; A)x\| \|R(\tau_0 - i\mu; A^*)y\| d\mu \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{\tau_n - iw}^{\tau_n - in} \frac{1}{t} e^{\lambda t} \langle R^2(\lambda; A)x, y \rangle d\lambda \right| \\ & \leq \frac{4}{t} e^{\tau_n t} \int_{-w}^{-n} \|R(\tau_0 + i\mu; A)x\| \|R(\tau_0 - i\mu; A^*)y\| d\mu. \end{aligned}$$

Therefore

$$\begin{aligned} & |\langle te^{-\tau_n t} (\exp(tA) - T_n(t) - T_n^{(1)}(t) - T_n^{(2)}(t))x, y \rangle| \\ & \leq \frac{2}{\pi} \int_{-\infty}^{+\infty} \|R(\tau_0 + i\mu; A)x\| \|R(\tau_0 - i\mu; A^*)y\| d\mu \\ & \leq N(x, y) \quad (\text{Lemma 1}) \end{aligned}$$

where $N(x, y)$ only depend on x and y . By the resonance theorem, there exists a positive constant number N_1 such that

$$\begin{aligned} & \|\exp(tA) - T_n(t) - T_n^{(1)}(t) - T_n^{(2)}(t)\| \\ (2.11) \quad & \leq \frac{1}{t} N_1 e^{\tau_n t}, \quad n = 1, 2, 3, \dots, \quad t > 0. \end{aligned}$$

From the definition of $T_n^{(1)}(t)$, $T_n^{(2)}(t)$, we get the estimations

$$\begin{aligned} \|T_n^{(1)}(t)\| & \leq \frac{1}{2\pi i} \left[\int_{\tau_n}^{\tau_0} e^{\tau t} (\|R(\tau + in; A)\| + \|R(\tau - in; A)\|) d\tau \right] \\ & \leq \frac{1}{\pi} [\|R(\tau_0 + in; A)\| + \|R(\tau_0 - in; A)\|] \int_{-\infty}^{\tau_0} e^{\tau t} d\tau \\ & \leq \frac{1}{\pi t_1} e^{\tau_0 t_2} [\|R(\tau_0 + in; A)\| + \|R(\tau_0 - in; A)\|] \quad \text{for } t_2 \geq t \geq t_1 \end{aligned}$$

and

$$\begin{aligned} \|T_n^{(2)}(t)\| & \leq \frac{1}{2\pi t} e^{\tau_n t} [\|R(\tau_n - in; A)\| + \|R(\tau_n + in; A)\|] \\ & \leq \frac{1}{\pi t_1} e^{\tau_n t_1} [\|R(\tau_0 - in; A)\| + \|R(\tau_0 + in; A)\|], \end{aligned}$$

so

$$\lim_{n \rightarrow \infty} \|T_n^{(1)}(t)\| = 0, \quad \lim_{n \rightarrow \infty} \|T_n^{(2)}(t)\| = 0.$$

Therefore from (2.11), we have

$$\lim_{n \rightarrow \infty} \|\exp(tA) - T_n(t)\| = 0$$

and the convergence is uniform on $t \in [t_1, t_2]$. Lemma 3 is proved. $\square$

Proof of Theorem 1. The “only if” part has been proved in [1, §2.3, Theorem 3.6]. We shall prove the “if” part. Let $t_0 > 0$ be some positive number. For an arbitrarily small positive number ε , we need only to prove that there exists a positive number δ , such that $\|\exp(t_0+h)A - \exp(t_0A)\| < \varepsilon$ for any h provided $|h| < \delta$.

By Lemma 3, there exists a positive integral N_0 such that $\|\exp(tA) - T_n(t)\| \leq \varepsilon/4$ for $n \geq N_0$ and $t \in [\frac{1}{2}t_0, \frac{3}{2}t_0]$. Therefore

$$\begin{aligned} \|\exp(t_0+h)A - \exp(t_0A)\| &\leq \|\exp(t_0+h)A - T_{N_0}(t_0+h)\| \\ &\quad + \|T_{N_0}(t_0+h) - T_{N_0}(t_0)\| \\ (2.12) \quad &\quad + \|T_{N_0}(t_0) - \exp(t_0A)\| \\ &\leq \frac{\varepsilon}{2} + \|T_{N_0}(t_0+h) - T_{N_0}(t_0)\| \quad \text{for } |h| \leq \frac{1}{2}t_0; \end{aligned}$$

but

$$\begin{aligned} \|T_{N_0}(t_0+h) - T_{N_0}(t_0)\| &\leq \frac{1}{2\pi} \int_{-N_0}^{N_0} e^{\tau_0 t_0} |e^{(\tau_0+iw)h} - 1| \cdot \|R(\tau_0+iw; A)\| dw \\ &\leq \frac{1}{\pi} N_0 e^{\tau_0 t_0} \psi(0) \cdot \max_{-N_0 \leq w \leq N_0} |e^{(\tau_0+iw)h} - 1|, \end{aligned}$$

noting

$$\lim_{h \rightarrow 0} |e^{(\tau_0+iw)h} - 1| = 0 \quad \text{uniformly for } w \in [-N_0, N_0].$$

So there exists some positive number δ , such that

$$(2.13) \quad \|T_{N_0}(t_0+h) - T_{N_0}(t_0)\| \leq \frac{\varepsilon}{2} \quad \text{for any } h \text{ provided } |h| < \delta.$$

Combining (2.12) and (2.13), Theorem 1 is proved. $\square$

REFERENCES

1. A. Pazy, *Semigroup of linear operators and application to partial differential equations*, Springer-Verlag, New York, Berlin, Heidelberg, and Tokyo, 1983.
2. F. L. Huang, *Characteristic conditions for exponential stability of linear dynamical systems in Hilbert space*, Ann. Differential Equations **1** (1985), 43–56.

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