SPECIAL α -LIMIT POINTS FOR MAPS OF THE INTERVAL

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ABSTRACT. The notion of a special α -limit point is defined. For maps of the interval, it is shown that a point is a special α -limit point if and only if it is an element of the attracting center.

1. INTRODUCTION

Our main result is that for continuous self-maps of an interval, the attracting center and the set of special α -limit points coincide. Also, combining the work in this paper with that of Xiong in [9], we will obtain a necessary and sufficient condition for the existence of a periodic point with period not a power of two.

Throughout this paper, f will be a continuous self-map of an interval I . For every positive integer n , f^n will denote the function f composed with itself n times. Using the forward orbit of a point x , the ω -limit set $\omega(x)$ of x is defined to be the collection of all limit points of the sequence $(f^n(x))_{n=1}^{\infty}$. Elements of $\omega(x)$ are called ω -limit points. A point z is said to be recurrent if $z \in \omega(z)$.

In [9] the α -limit set $\alpha(x)$ is defined for self-maps of the interval. In this paper we study the special α -limit set of a point x and denote it by $s\alpha(x)$. A point y is an element of the set $s\alpha(x)$ provided there exists a sequence of positive integers $(n(i))_{i=1}^{\infty}$ and a sequence of points $(y_i)_{i=0}^{\infty}$ such that

- (1) $x = y_0$,
- (2) $f^{n(i)}(y_i) = y_{i-1}$,
- (3) $\lim_{i \rightarrow \infty} y_i = y$.

Note that if such a sequence exists then y is an element of $s\alpha(y_i)$ for every i .

In order to state our theorem, we must define what is called the attracting center of a dynamical system. For a subset Y of I , define $\Lambda(Y) = \bigcup_{x \in Y} \omega(x)$. Let $\Lambda^1 = \Lambda(I)$. For every $n > 1$, inductively define $\Lambda^n = \Lambda(\Lambda^{n-1})$. The attracting center Λ^∞ is then the intersection of the sets Λ^n .

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Theorem. Suppose that f is a continuous self-map of an interval I . Then the following are equivalent:

- (1) $x \in s\alpha(y)$ for some y ,
- (2) $x \in s\alpha(x)$,
- (3) $x \in \Lambda^\infty$.

Combining this theorem with the work of Xiong [9], we obtain the following corollary. The period of a periodic point x is the least n such that $f^n(x) = x$.

Corollary. The following are equivalent:

- (1) Some point y that is not recurrent is a special α -limit point.
- (2) Some periodic point has period that is not a power of two.

A weaker form of recurrence than that of being an ω -limit point is that of being a nonwandering point. Let $y \in I$. Then y is a nonwandering point if for every open neighborhood V_y of y , $V_y \cap f^m(V_y) \neq \emptyset$ for some m .

A pointwise definition of the set of nonwandering points can be obtained by considering the backwards orbit of a point. A point y is an element of $\alpha(x)$ provided that a strictly increasing sequence of positive integers $(n(i))_{i=1}^\infty$ and a sequence of points $(y_i)_{i=1}^\infty$ can be found such that

- (1) $f^{n(i)}(y_i) = x$,
- (2) $\lim_{i \rightarrow \infty} y_i = y$.

Elements of $\alpha(x)$ are called α -limit points. The sets of all α -limit points, in general, strictly contains the set of nonwandering points. A pointwise definition of the set Ω of nonwandering points, however, can be obtained using

Proposition A [6]. If f is a self-map of an interval I then $x \in \Omega$ if and only if $x \in \alpha(x)$.

We now provide an example of a self-map of $[0, 4]$ for which $\alpha(1) \neq s\alpha(1)$. A smooth version of this example is used in [7] to show that the point 1 is a nonwandering point but not an ω -limit point. We will see later that if y is a nonwandering point that is not an ω -limit point then $y \in \alpha(y) - s\alpha(y)$. According to the theorem, when $\Lambda^1 \neq \Lambda^2$ an ω -limit point y can be found with $y \in \alpha(y) - s\alpha(y)$.

Example.

$$f(x) = \begin{cases} x + 2 & \text{if } x \in [0, 1), \\ 3x & \text{if } x \in [1, \frac{4}{3}), \\ -\frac{9}{2}x + 10 & \text{if } x \in [\frac{4}{3}, 2), \\ 2x - 3 & \text{if } x \in [2, \frac{7}{2}), \\ 4 & \text{if } x \in [\frac{7}{2}, 4]. \end{cases}$$

It is clear from the graph of the function f that there is a strictly increasing sequence of points $(w_i)_{i=1}^\infty$ with $w_1 = 2$, $w_i \in (2, 3)$ and $f(w_i) = w_{i-1}$ for $i > 1$. For every $i \geq 1$ there is a $z_i \in [0, 1]$ with $f(z_i) = w_i$. Thus $f^{i+1}(z_i) = 1$ and $\lim_{i \rightarrow \infty} z_i = 1$, hence $1 \in \alpha(1)$. The point 1, however, cannot be a special α -limit point since $f^n([1, \frac{14}{9}]) = [3, 4]$ for every $n > 0$ and $f^{-1}([0, 1)) = \emptyset$. To see this, one needs only to realize that if 1 were a special α -limit point, we could find points y_1 , y_2 , and y_3 all elements of $(0, \frac{14}{9})$ and integers $n(3)$ and $n(2)$ with $f^{n(3)}(y_3) = y_2$ and $f^{n(2)}(y_2) = y_1$.

2. PRELIMINARIES

A point y is said to be a γ -limit point of x if $y \in \omega(x) \cap \alpha(x)$. Let $\gamma(x) = \omega(x) \cap \alpha(x)$ and $\Gamma = \bigcup_{x \in I} \gamma(x)$. It follows from Proposition A that the set of recurrent points R is contained in the set Γ . In his investigation of the set Λ^∞ , Xiong [9] proves the following theorem.

Theorem A. *Suppose that f is a continuous self-map of an interval. Then*

- (1) $\Gamma = \Lambda^2 = \Lambda^\infty \subseteq \overline{P}$, where P is the set of periodic points;
- (2) $R = \Gamma$ if and only if the period of every periodic point is a power of 2, where R is the set of all recurrent points.

A function f displays a horseshoe if there are disjoint closed intervals J and K in the interval and a positive integer n such that $f^n(J) \supset J$, $f^n(K) \supset K$, $f^n(K) \supset J$, and $f^n(J) \supset K$. The content of statement (2) of Theorem A is that Γ strictly contains R if and only if f displays a horseshoe. In general, the following set relation holds for self-maps of the interval

$$P \subseteq R \subseteq \Gamma \subseteq \overline{P} \subseteq \Lambda^1 \subseteq \Omega.$$

In the above example, the set Λ^1 is strictly contained in Ω . For an example of a function where $\overline{P}$ is strictly contained in Λ^1 see [2]. In [3] an example of a function with $\Gamma \neq \overline{P}$ is constructed.

3. PROOF OF THE THEOREM

We will prove several lemmas from which the theorem will follow immediately.

Lemma 1. *If $x \in s\alpha(y)$ then $f^n(x) \in s\alpha(y)$ for every n and $\omega(x) \subseteq s\alpha(y)$.*

Proof. Suppose that $x \in s\alpha(y)$. Then we can find y_i with $\lim_{i \rightarrow \infty} y_i = x$ and $n(i) > 0$ such that $f^{n(i)}(y_i) = y_{i-1}$ and $f^{n(1)}(y_1) = y$.

Consider the sequence

$$\begin{aligned} & (y, f^{n(1)-1}(y_1), \dots, y_1, f^{n(2)-1}(y_2), \dots, y_2, f^{n(3)-1}(y_3), \dots) \\ & \equiv (y, z_1, z_2, z_3, \dots). \end{aligned}$$

Note that if $m < \sum_{i=1}^j n(i)$ then $f^m(y_j)$ is a term of the sequence $(z_i)_{i=1}^\infty$. One can see that any limit point of the sequence $(z_i)_{i=1}^\infty$ is an element of $s\alpha(y)$.

Let $z \in \{f(x), f^2(x), \dots\} \cup \omega(x)$ and let $\varepsilon > 0$ be given. Then we can find an $m > 0$ with $f^m(x) \in (z - \varepsilon, z + \varepsilon)$. The function f^m is continuous and $\lim_{i \rightarrow \infty} y_i = x$, thus for some $N > 0$ if $j > N$ then $f^m(y_j) \in (z - \varepsilon, z + \varepsilon)$. If we take $j > \max(N, m)$ then $m < j \leq \sum_{i=1}^j n(i)$. Thus $f^m(y_j) \in (z - \varepsilon, z + \varepsilon)$ and is a term of the sequence $(z_i)_{i=1}^\infty$. It follows that z is a limit point of the sequence $(z_i)_{i=1}^\infty$. $\square$

To state Lemma 2, we develop some notation. We will use the symbols $\alpha_R(x)$ (resp., $\alpha_L(x)$) to denote the set of all points y such that there exists a sequences $(n(i))_{i=1}^\infty$ and $(x_i)_{i=1}^\infty$ with $\lim_{i \rightarrow \infty} n(i) = \infty$, $\lim_{i \rightarrow \infty} x_i = y$, $f^{n(i)}(x_i) = x$, and $y < x_i$ (resp., $x_i < y$) for every $i > 0$.

Lemma 2. (1) *If for some $\varepsilon > 0$, $x \in \alpha_L(y)$ for every $y \in (x - \varepsilon, x)$, then for every $y \in (x - \varepsilon, x)$, $x \in s\alpha(y)$.*

(2) If for some $\varepsilon > 0$, $x \in \alpha_R(y)$ for every $y \in (x, x + \varepsilon)$, then for every $y \in (x, x + \varepsilon)$, $x \in s\alpha(y)$.

Proof. We will prove (1). Let $y \in (x - \varepsilon, x)$. Since $x \in \alpha_L(y)$, we can find $y_1 \in (x - \varepsilon/2, x)$ and $n(1) > 0$, with $f^{n(1)}(y_1) = y$. By hypothesis, $x \in \alpha_L(y_1)$; thus we can find $y_2 \in (x - \varepsilon/2^2, x)$ and $n(2) > 0$, with $f^{n(2)}(y_2) = y_1$. Continuing in this way, we obtain a sequence $(y_i)_{i=1}^\infty$ and $n(i) > 0$, with $y_i \in (x - \varepsilon/2^i, x)$ and $f^{n(i)}(y_i) = y_{i-1}$. $\square$

Before proceeding any further, we need the definitions of almost periodic points and minimal sets. If given an open set V_y containing y , one can find an integer $n > 0$ such that for any integer $q > 0$ there is an integer r , $q \leq r \leq q + n$ with $f^r(y) \in V_y$, then y is said to be almost periodic. If for every $x \in \omega(y)$ we have that $\omega(x) = \omega(y)$, then $\omega(y)$ is said to be a minimal set. A well-known fact is that for compact metric spaces a point y is almost periodic if and only if $y \in \omega(y)$ and $\omega(y)$ is a minimal set.

Lemma 3. If x is almost periodic then $x \in s\alpha(x)$.

Proof. Since x is almost periodic, $x \in \omega(x)$ and $\omega(x)$ is a minimal set. Since $f(\omega(x)) = \omega(x)$, we can find a sequence of points $(z_i)_{i=1}^\infty$ with $z_i \in \omega(x)$, $f(z_i) = z_{i-1}$, and $f(z_1) = x$. Let y be a limit point of this sequence. Then $y \in s\alpha(x)$ and, since $\omega(x)$ is a closed set, $y \in \omega(x)$. Since in this case $\omega(x)$ is a minimal set, $x \in \omega(y)$. It follows from Lemma 1 that $x \in s\alpha(x)$. $\square$

Let Y be a subset of I . We will denote the closure of Y by $\overline{Y}$. A point z is called a right-side (resp., left-side) accumulation point of Y if for every $\varepsilon > 0$, $(z, z + \varepsilon) \cap Y \neq \emptyset$ (resp., $(z - \varepsilon, z) \cap Y \neq \emptyset$). The right-side closure $\overline{Y}_R$ (resp., left-side closure $\overline{Y}_L$) is the union of Y and the set of right-side (resp., left-side) accumulation points of Y .

In Lemma 4 bilateral limit points of periodic points are considered. We will see that such points are special α -limit points. If in addition the point x in question is not almost periodic, then we show that x is the end point of interval J with $x \in s\alpha(y)$ for every $y \in J$. In either case we will see that $x \in s\alpha(x)$.

Lemma 4. If $x \in \overline{P}_L \cap \overline{P}_R$ then $x \in s\alpha(x)$. If in addition x is not almost periodic, then for some $\varepsilon > 0$, either $x \in s\alpha(y)$ for every $y \in (x - \varepsilon, x]$, or $x \in s\alpha(y)$ for every $y \in [x, x + \varepsilon)$.

Proof. According to Lemma 3, we may assume that x is not almost periodic. If for every $\varepsilon > 0$ we could find an $n > 0$ such that $f^{nk}(x) \in (x - \varepsilon, x + \varepsilon)$ for every k , then x would be almost periodic. Thus we can find an $\varepsilon > 0$ such that if $n > 0$ is fixed then for some $k > 0$, $f^{nk}(x) \notin (x - \varepsilon, x + \varepsilon)$.

Fix such an $\varepsilon > 0$. By our hypothesis, $\lim_{i \rightarrow \infty} p_i = x$, $\lim_{i \rightarrow \infty} q_i = x$ with $p_i < x$ and $x < q_i$ for two sequences of periodic points $(p_i)_{i=1}^\infty$ and $(q_i)_{i=1}^\infty$. Furthermore, we may assume that $p_i \in (x - \varepsilon, x)$ and $q_i \in (x, x + \varepsilon)$. Let $n(i)$ be the period of p_i and $m(i)$ the period of q_i . Let $l(i) = n(i)m(i)$. It follows from our choice of ε that we can find $k(i) > 0$ with $f^{l(i)k(i)}(x) \notin (x - \varepsilon, x + \varepsilon)$.

Suppose by choosing a subsequence, if needed, that $f^{l(i)k(i)}(x) < x - \varepsilon$ for every i . Since the period of p_i is $n(i)$, $f^{l(i)k(i)}(p_i) = p_i$; thus

$$(*) \quad (x - \varepsilon, p_i] \subseteq f^{l(i)k(i)}([p_i, x)).$$

Let $y \in (x - \varepsilon, x)$. We will now show that $x \in s\alpha(y)$. Since $\lim_{i \rightarrow \infty} p_i = x$, we can find a positive integer $i(1)$ such that $x - \varepsilon < y < p_{i(1)} < x$. It follows

from (*) that there exists $z_{i(1)} \in (p_{i(1)}, x)$ with $f^{l(i(1))k(i(1))}(z_{i(1)}) = y$. Since $z_{i(1)} \in (p_{i(1)}, x)$, we can find a positive integer $i(2)$ with

$$x - \varepsilon < y < p_{i(1)} < z_{i(1)} < p_{i(2)} < x.$$

It follows from (*) that there exists $z_{i(2)} \in (p_{i(2)}, x)$ with $f^{l(i(2))k(i(2))}(z_{i(2)}) = z_{i(1)}$. Continuing in this way, it is possible to find $i(j)$, $p_{i(j)}$, and $z_{i(j)}$ with

$$x - \varepsilon < y < p_{i(1)} < z_{i(1)} < p_{i(2)} < \cdots < p_{i(j)} < z_{i(j)} < \cdots < x$$

such that

$$f^{l(i(j))k(i(j))}(z_{i(j)}) = z_{i(j-1)}.$$

Since $\lim_{i \rightarrow \infty} p_i = x$, we have that $\lim_{j \rightarrow \infty} z_{i(j)} = x$. Thus $x \in s\alpha(y)$.

To finish the proof of the lemma it suffices to show that we can find a point $z \in (x - \varepsilon, x)$ and an $n > 0$ with $f^n(z) = x$. The argument that we will give here is taken from the proof of Proposition C in [9]. Let $g = f^{n(1)}$ and $L = [p_1, x]$. Then since $g(p_1) = p_1$, $K = L \cup g(L) \cup g^2(L) \cup \cdots$ is an interval. Let $l(j)$ denote the period of p_j with respect to g . For $k = 1, 2$, or 3 suppose that a subsequence of $g^{l(1)-k}(p_1)g^{l(2)-k}(p_2) \cdots g^{l(j)-k}(p_j) \cdots$ converges to $u_k \in \bar{K}$. It is clear that $g^k(u_k) = x$. If $u_{k'} = u_{k''}$ for some $k' < k''$, then

$$g^{k''-k'}(x) = g^{k''-k'}(g^{k'}(u_{k'})) = g^{k''}(u_{k'}) = g^{k''}(u_{k''}) = x.$$

However, x is by assumption not periodic, thus we must have that u_1, u_2 , and u_3 are distinct points in $\bar{K}$. Since K is an interval, $u_k \in K$ for some k . Hence for some point $z \in L$ and $n \geq 0$, $g^n(z) = u_k$. Thus $g^{n+k}(z) = x$. $\square$

Note that the last argument in the proof of Lemma 4 shows that for points $x \in \bar{P}_L - P$ (resp., $\bar{P}_R - P$), we have $x \in \alpha_L(x)$ (resp., $\alpha_R(x)$). We will use the symbols $\omega_R(x)$ (resp., $\omega_L(x)$) to denote the set of right-side (resp., left-side) accumulation points of the set $\{f^n(x)\}_{n=1}^\infty$.

Lemma 5. *If $x \in R$ then $x \in s\alpha(x)$.*

Proof. Let $x \in R$. If $\omega(x)$ is a minimal set, then according Lemma 3, $x \in s\alpha(x)$.

We assume that $\omega(x)$ is not a minimal set. Applying Lemma 3 of [1] to the compact set $\omega(x)$, we have that the set $A = \{y | \omega(y) = \omega(x)\} \cap \omega(x)$ is an uncountable set. Let Δ denote the symmetric difference of sets. Since $\bar{P}_L \Delta \bar{P}_R$ and $\omega_L(x) \Delta \omega_R(x)$ are both countable, $\bar{P}_L \cap \bar{P}_R$ (resp., $\omega_L(x) \cap \omega_R(x)$) is the complement of a countable set in $\bar{P}$ (resp., $\omega(x)$); since $\omega(x) \subseteq \bar{P}$, it follows that

$$\omega_L(x) \cap \omega_R(x) \cap \bar{P}_L \cap \bar{P}_R$$

is the complement of a countable set in $\omega(x)$. But then A , being an uncountable subset of $\omega(x)$, must intersect the set $\omega_L(x) \cap \omega_R(x) \cap \bar{P}_L \cap \bar{P}_R$. Let y be a point from A in this intersection.

Since $\omega(x) = \omega(y)$ and $\omega(x)$ is not a minimal set, it must be that y is not almost periodic. Without loss of generality, by Lemma 4 we may choose an $\varepsilon > 0$ such that if $z \in (y - \varepsilon, y]$ then $y \in s\alpha(z)$. Now $y \in \omega_L(x)$; hence for some positive integer n , $f^n(x) \in (y - \varepsilon, y)$. Since the function f^n is continuous, for some $\delta > 0$

$$(**) \quad f^n((x - \delta, x + \delta)) \subseteq (y - \varepsilon, y).$$

Now $R \subseteq \Omega$, so by Proposition A, $x \in \alpha(x)$. Thus for some $z \in (x - \delta, x + \delta)$ and $m > n$, $f^m(z) = x$. By (**), $f^n(z) \in (y - \varepsilon, y)$, hence $y \in s\alpha(f^n(z))$ and $f^{m-n}(f^n(z)) = x$. It follows that $y \in s\alpha(x)$. Since $x \in \omega(y)$, by Lemma 1, $x \in s\alpha(x)$. $\square$

We consolidate our proof that (1) implies (2) implies (3) in our theorem into the proof of Proposition 1. Before we prove Proposition 1, two well-known results must be stated. The first, Lemma B is a description of the dynamical behavior that can occur on subintervals of I containing no periodic points. The second, Proposition C, is a description of a situation under which one can conclude that a point is an element of $\omega_R(x)$ (resp., $\omega_L(x)$) for some x . For a proof of Lemma B see [5] and for Proposition C see [4]. Proposition C is due to Sharkovskii in [8].

Lemma B. *Let J be a subinterval of I that does not contain any periodic points. If x and y are elements of J and if for some n and m both $f^n(x)$ and $f^m(y)$ are elements of J , then either $f^n(x) < x$ and $f^m(y) < y$, or $f^n(x) > x$ and $f^m(y) > y$.*

Proposition C. *If for every $\varepsilon > 0$, an $n > 0$ can be found with*

$$(c, c + \varepsilon) \cap f^n((c, c + \varepsilon)) \neq \emptyset$$

(resp., $(c - \varepsilon, c) \cap f^n((c - \varepsilon, c)) \neq \emptyset$), then $c \in \omega_R(y) \cup P$ (resp., $c \in \omega_L(y) \cup P$) for some $y \in I$.

Proposition 1. *If $x \in s\alpha(y)$ for some y then $x \in s\alpha(x)$ and $x \in \Gamma$.*

Proof. Let $x \in s\alpha(y)$. Since $R \subseteq \Gamma$, by Lemma 5 we may assume that x is not recurrent.

Fix an $\varepsilon > 0$ such that $f^n(x) \notin (x - \varepsilon, x + \varepsilon)$ for every $n > 0$. As $x \in s\alpha(y)$ we can find a sequence $(y_i)_{i=1}^\infty$ and $n(i) > 0$ with $\lim_{i \rightarrow \infty} y_i = x$ and $f^{n(i)}(y_i) = y_{i-1}$. By choosing a subsequence of the sequence $(y_i)_{i=1}^\infty$, we may assume that $y_i \in (x - \varepsilon, x)$ for every $i > 0$.

Claim 1. The point x is an element of $\overline{P}_L - P$ and hence, as noted above, an element of $\alpha_L(x)$.

Proof of Claim 1. Let $\delta > 0$ be given. It is possible to find a $j > 0$ with y_j and y_{j-1} both elements of $(x - \delta, x)$ and hence, $(x - \delta, x) \cap f^{n(j)}((x - \delta, x)) \neq \emptyset$. Since x is not periodic by Proposition C, $x \in \omega_L(y)$ for some y . Now if, on the contrary, $x \notin \overline{P}_L$ then we can find an $\bar{\varepsilon} > 0$ such that $(x - \bar{\varepsilon}, x) \cap P = \emptyset$. Since $x \in \omega_L(y)$, we can assume that $y \in (x - \varepsilon, x)$. Thus for some $n > 0$, $x - \varepsilon < y < f^n(y) < x$. Since $\lim_{i \rightarrow \infty} y_i = x$, we can find an integer j with $x - \bar{\varepsilon} < y_{j-1} < y_j < x$. This is, however, a contradiction to Lemma B as $f^{n(j)}(y_j) = y_{j-1}$. Thus $x \in \overline{P}_L - P$ and hence $x \in \alpha_L(x)$.

Claim 2. For some $\delta > 0$, $x \in s\alpha(y)$ for every $y \in (x - \delta, x]$.

Proof of Claim 2. To prove this claim we first define the integers $M(k, j)$ for every $k > 0$ and $j > 0$, according to the equation $M(k, j) = \sum_{i=1}^j n(k + i)$. Since $f^{n(i)}(y_i) = y_{i-1}$, we have that $f^{M(k, j)}(y_{k+j}) = y_k$. We consider two cases.

Case 1. For some $k > 0$, $x + \varepsilon < f^{M(k,j)}(x)$ for an infinite number of j . Fix such a $k > 0$. Let $(h(i))_{i=1}^{\infty}$ be an increasing subsequence of positive integers such that $x < f^{M(k,h(i))}(x)$ for every $i > 0$. In this case we will show that for every $y \in (y_k, x]$, $x \in s\alpha(y)$. Since $x \in \alpha_L(x)$, by Lemma 2 it suffices to show that $x \in \alpha_L(y)$ for every $y \in (y_k, x)$. So let $y \in (y_k, x)$. For every $i > 0$,

$$[y_k, f^{M(k,h(i))}(x)] \subseteq f^{M(k,h(i))}([y_{k+h(i)}, x]).$$

Hence for every $i > 0$ we can find a point $z_i \in [y_{k+h(i)}, x]$ with $f^{M(k,h(i))}(z_i) = y$. Since $\lim_{i \rightarrow \infty} y_{k+h(i)} = x$, it follows that $x \in \alpha_L(y)$.

Case 2. For every $k > 0$, we can find a positive integer $M(k)$ so that if $j > M(k)$ then $f^{M(k,j)}(x) < x - \varepsilon$. In this case we will prove that for $y \in (x - \varepsilon, x)$, $x \in \alpha_L(y)$. Let $y \in (x - \varepsilon, x)$. We can find a $k > 0$ with $y < y_k$. By our assumption for $j > M(k)$, we have

$$f^{M(k,j)}(x) < x - \varepsilon < y < y_k.$$

It follows that for every $j > M(k)$ we can find a $z_j \in [y_{k+j}, x]$ with $f^{M(k,j)}(z_j) = y$. As in Case 1, since $\lim_{j \rightarrow \infty} y_{k+j} = x$, $x \in \alpha_L(y)$. By Lemma 2 and the fact that $x \in \alpha_L(x)$, we see that for every $y \in (x - \varepsilon, x]$, $x \in s\alpha(y)$. This completes the proof of the claim.

To complete the proof of the lemma we must show that $x \in \Gamma$. Let $\delta > 0$ be such that for every $y \in (x - \delta, x)$, $x \in s\alpha(y)$. According to the proof of Claim 1, $x \in \omega_L(y)$ for some $y \in (x - \delta, x)$, and hence $x \in \omega_L(y) \cap s\alpha(y)$. Since $s\alpha(y) \subseteq \alpha(y)$, $x \in \Gamma$.

The following lemma will complete the proof of the theorem.

Lemma 6. *If $x \in \Gamma$ then $x \in s\alpha(x)$.*

Proof. If $x \in R$ then, by Lemma 5, $x \in s\alpha(x)$. According to Xiong in [9],

$$\Gamma = \left(\bigcup_{y \in I} \omega_L(y) \cap \alpha_L(y) \right) \cup \left(\bigcup_{y \in I} \omega_R(y) \cap \alpha_R(y) \right) \cup P.$$

Thus we assume that $x \notin R$ and that y has been chosen with $x \in \omega_L(y) \cap \alpha_L(y)$. Let $\varepsilon > 0$ be chosen so that if $n > 0$ then $f^n(x) \notin (x - \varepsilon, x + \varepsilon)$. Let $(y_i)_{i=1}^{\infty}$ and $(z_i)_{i=1}^{\infty}$ be sequences with $y_i \in (x - \varepsilon, x)$, $z_i \in (x - \varepsilon, x)$, $f^{n(i)}(y) = y_i$, $f^{m(i)}(z_i) = y$, $\lim_{i \rightarrow \infty} n(i) = \infty$, and $\lim_{i \rightarrow \infty} m(i) = \infty$. Let $L(i, j) = m(i) + n(j)$. Then $f^{L(i,j)}(z_i) = y_j$ for all $i > 0$ and $j > 0$.

Case 1. There exists $j > 0$ such that $f^{L(i,j)}(x) > x + \varepsilon$ for an infinite number of i . Let $(h(i))_{i=1}^{\infty}$ be an increasing subsequence of positive integers such that $f^{L(h(i),j)}(x) > x + \varepsilon$. Since $f^{L(h(i),j)}(z_{h(i)}) = y_j$, it follows that

$$(***) \quad [y_j, x + \varepsilon] \subseteq f^{L(h(i),j)}([z_{h(i)}, x]).$$

To finish the proof by Lemmas 2 and Proposition 1, it suffices to show that for every $y \in (y_j, x)$, $x \in \alpha_L(y)$.

Let $y \in (y_j, x)$. For every $i > 0$, there exists $w_i \in (z_{h(i)}, x)$ with $f^{L(h(i),j)}(w_i) = y$. It follows that $x \in \alpha_L(y)$.

Case 2. If $j > 0$ then there exists an integer $M(j)$ such that for $i > M(j)$, $f^{L(i,j)}(x) < x - \varepsilon$.

To finish the proof in this case, we will show that if $y \in (x - \varepsilon, x)$ then, $x \in \alpha_L(y)$. Let $y \in (x - \varepsilon, x)$. Since $\lim_{i \rightarrow \infty} y_i = x$, we can find a $j > 0$ with $x - \varepsilon > y < y_j$. For every $i > M(j)$ we have

$$(x - \varepsilon, y_j] \subseteq f^{L(i,j)}([z_i, x)).$$

Hence for every $i > M(j)$, we can find $w_i \in (z_i, x)$ with $f^{L(i,j)}(w_i) = y$. It follows that $x \in \alpha_L(y)$. $\square$

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