NOETHER NORMALIZATIONS FOR LOCAL RINGS OF ALGEBRAIC VARIETIES

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ABSTRACT. When D is a regular subring of A such that the inclusion map is finite, D is called a Noether normalization of A . We will prove the existence of Noether normalizations of A , when A is a local ring of a one-dimensional algebraic variety. Furthermore we will give a criterion for the existence and interesting examples.

1. INTRODUCTION

Consider the following problem.

Problem. Let K be a field and (A, m) a local domain essentially of finite type over K . Then, does there exist a regular subring D of A such that the inclusion map $D \hookrightarrow A$ is a finite morphism?

If such D exists, it is called a *Noether normalization* of A .

It is well known that affine domains or complete local domains have Noether normalizations. So it seems to be natural to ask whether local rings of algebraic varieties have Noether normalizations or not.

In the next section we will prove that there exists a Noether normalization if the given ring A has dimension 1. Furthermore, in §3 some sufficient conditions for the existence of Noether normalizations will be given. The final section is devoted to some examples.

The author does not know whether local rings of algebraic varieties always have Noether normalizations or not.

2. THE CASE OF DIMENSION 1

Throughout this paper, K denotes a field and (A, m) a local domain essentially of finite type over K , i.e., a localization of an affine domain over K .

Definition 2.1. A regular subring D of A is called a *Noether normalization* when the inclusion map $D \hookrightarrow A$ is (module) finite.

Remark 2.2. When we show the existence of Noether normalizations of A , we

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may assume that the residue class field $A/\mathfrak{m}$ is a finite algebraic extension over K and A is a local ring of a hypersurface.

If $(A, \mathfrak{m})$ has a Noether normalization D , D must be a regular local ring and have the same dimension as A .

Next the existence of Noether normalization will be proved in the case of $\dim A = 1$.

Theorem 2.3. *Let A be a local domain of a closed point of a one-dimensional algebraic variety over K . Then A has a Noether normalization.*

Proof. Let X be a one-dimensional projective variety over K and $x \in X$ a closed point such that $\mathcal{O}_{x,X} = A$. When we set $\pi: \tilde{X} \rightarrow X$ as the normalization, $\tilde{X}$ is a regular projective variety over K . Put $\pi^{-1}(x) = \{y_1, \dots, y_l\}$ and $B = \bigcap_{i=1}^l \mathcal{O}_{y_i, \tilde{X}}$. Hence B is a semilocal Dedekind domain and so a principal ideal domain. Let $\{\mathfrak{m}_1, \dots, \mathfrak{m}_l\}$ be the set of maximal ideals of B and set $B_{\mathfrak{m}_i} = \mathcal{O}_{y_i, \tilde{X}}$. Then there exists $t \in K(X) = K(\tilde{X})$ (the function field) such that $\{y_1, \dots, y_l\}$ is just equal to the set of zeros of the rational function t on $\tilde{X}$. (See Hartshorne [1, Chapter IV].) Since $\mathcal{M}_{y_i, \tilde{X}}$ (the maximal ideal of $\mathcal{O}_{y_i, \tilde{X}}$) contains t for every i , the polynomial ring $K[t]$ is the subring of B . Furthermore B contains the localization $K[t]_{(t)}$ because t is contained in the Jacobson radical of B . Note that $\{B_{\mathfrak{m}_1}, \dots, B_{\mathfrak{m}_l}\}$ is just the set of discrete valuation rings of $K(\tilde{X})$ dominating $K[t]_{(t)}$. Then the inclusion map $K[t]_{(t)} \hookrightarrow B$ is finite because the integral closure of $K[t]_{(t)}$ in $K(\tilde{X})$ must coincide with the intersection of all discrete valuation rings of $K(\tilde{X})$ dominating $K[t]_{(t)}$.

Since B is a principal ideal domain, there exists a prime element t_i in B such that $\mathfrak{m}_i = (t_i)$ for each i . So we can describe the conductor ideal of $\mathcal{O}_{x,X}$ in B as $\mathcal{C}_{B/\mathcal{O}_{x,X}} = (t_1^{a_1} \cdots t_l^{a_l})$. Then for a sufficiently large n , we get $t^n \in \mathcal{C}_{B/\mathcal{O}_{x,X}} \subseteq \mathcal{M}_{x,X}$. Consider the following diagram:

$$\begin{array}{ccc} \mathcal{O}_{x,X} & \longrightarrow & B \\ \uparrow & & \uparrow \\ K[t^n]_{(t^n)} & \longrightarrow & K[t]_{(t)} \end{array}$$

Since both $K[t^n]_{(t^n)} \rightarrow K[t]_{(t)}$ and $K[t]_{(t)} \rightarrow B$ are finite, so is the composite map $K[t^n]_{(t^n)} \rightarrow B$. So we have got a Noether normalization $K[t^n]_{(t^n)} \rightarrow \mathcal{O}_{x,X}$. Q.E.D.

3. THE CASE OF DIMENSION ≥ 2

This section is devoted to proving the following theorem.

Theorem 3.1. *Let X be an n -dimensional projective variety over a field K and $x \in X$ a closed point of X . Then the following are equivalent.*

(1) *There exists a system of parameters $t_1, \dots, t_n$ of $\mathcal{O}_{x,X}$ such that the inclusion map*

$$K[t_1, \dots, t_n]_{(t_1, \dots, t_n)} \hookrightarrow \mathcal{O}_{x,X}$$

is finite.

(2) *There exists a system of parameters $t_1, \dots, t_n$ of $\mathcal{O}_{x,X}$ such that*

$$[K(X): K(t_1, \dots, t_n)] = e_{(t_1, \dots, t_n)\mathcal{O}_{x,X}}(\mathcal{O}_{x,X}) \cdot [\mathcal{O}_{x,X}/\mathcal{M}_{x,X}: K].$$

(3) *There exists an n -dimensional subvariety V of $X \times \mathbf{A}^n$ and a closed point y of V such that $p_1(y) = x$, $\mathcal{O}_{x,X} \xrightarrow{\sim} \mathcal{O}_{y,V}$, $p_2(y) = (0, \dots, 0) \in \mathbf{A}^n$, and $p_2^{-1}((0, \dots, 0)) \cap V = \{y\}$, where $\mathbf{A}^n$ denotes the n -dimensional affine space over K , $p_1: X \times \mathbf{A}^n \rightarrow X$ is the first projection, and $p_2: X \times \mathbf{A}^n \rightarrow \mathbf{A}^n$ is the second one, respectively.*

Furthermore if X is a rational surface over $K = \overline{\mathbb{Q}}$, i.e., the algebraic closure of the rational number field $\mathbb{Q}$, the following is equivalent to the conditions as above.

(4) $\mathcal{O}_{x,X}$ has a Noether normalization.

Proof. At first we will prove (1) $\Leftrightarrow$ (2). That (1) implies (2) is an immediate consequence of Corollary 1 in [5, p. 299]. Conversely assume (2). Put $D = K[t_1, \dots, t_n]_{(t_1, \dots, t_n)}$. Let C be the integral closure of D in $\mathcal{O}_{x,X}$. Denote by $\{\mathfrak{m}_1, \dots, \mathfrak{m}_l\}$ the set of maximal ideals of C . Since $\mathcal{M}_{x,X} \cap D = (t_1, \dots, t_n)$, $\mathcal{M}_{x,X} \cap C$ is a maximal ideal of C . Suppose $\mathcal{M}_{x,X} \cap C = \mathfrak{m}_1$. Then by Zariski's Main Theorem [3, Chapitre IV], we get $C_{\mathfrak{m}_1} = \mathcal{O}_{x,X}$. On the other hand we have

$$\begin{aligned} [K(X): K(t_1, \dots, t_n)] &= [Q(C): Q(D)] = \text{rank}_D C \\ &= \text{rank}_D C \cdot e_{(t_1, \dots, t_n)D}(D) = e_{(t_1, \dots, t_n)D}(C) \\ &= \sum_{i=1}^l e_{(t_1, \dots, t_n)C_{\mathfrak{m}_i}}(C_{\mathfrak{m}_i}) \cdot [C_{\mathfrak{m}_i}/\mathfrak{m}_i C_{\mathfrak{m}_i}: K] \\ &= e_{(t_1, \dots, t_n)\mathcal{O}_{x,X}}(\mathcal{O}_{x,X}) \cdot [\mathcal{O}_{x,X}/\mathcal{M}_{x,X}: K], \end{aligned}$$

where $Q(*)$ means the field of quotients. Hence C must be a local ring. Therefore C coincides with $\mathcal{O}_{x,X}$ and D is a Noether normalization of $\mathcal{O}_{x,X}$.

Next we will show (1) $\Leftrightarrow$ (3). Assume that (3) is satisfied. Denote by f the composite map $V \hookrightarrow X \times \mathbf{A}^n \xrightarrow{p_2} \mathbf{A}^n$. Then there exists an open subscheme U of $\mathbf{A}^n$ containing the origin $(0, \dots, 0)$ such that $f^{-1}(u)$ is a finite set for any u contained in U . Since the morphism $f|_{f^{-1}(U)}: f^{-1}(U) \rightarrow U$ is projective and quasi-finite, i.e., any fibre is a finite set, $f|_{f^{-1}(U)}$ is a finite morphism. So $\mathcal{O}_{(0), \mathbf{A}^n} \rightarrow \mathcal{O}_{y,V} \simeq \mathcal{O}_{x,X}$ is finite.

Conversely assume that $K[t_1, \dots, t_n]_{(t_1, \dots, t_n)} \xrightarrow{f} \mathcal{O}_{x,X}$ is finite. Let $\text{Spec}(A)$ be an affine open set of X containing x such that A contains all $f(t_i)$'s. Denote by $\mathfrak{m}$ the maximal ideal of A corresponding to x . Note that $\mathfrak{m}$ contains $f(t_i)$ for every i .

$$\begin{aligned} X \times \mathbf{A}^n &\supset \text{Spec}(A) \times \text{Spec}(K[x_1, \dots, x_n]) = \text{Spec}(A[x_1, \dots, x_n]) \\ &\hookrightarrow \text{Spec}(A[x_1, \dots, x_n]/(x_1 - f(t_1), \dots, x_n - f(t_n))) \simeq \text{Spec}(A). \end{aligned}$$

Put $V^\circ = \text{Spec}(A[x_1, \dots, x_n]/(x_1 - f(t_1), \dots, x_n - f(t_n)))$. Then it is obvious that V° is an n -dimensional closed subvariety of $\text{Spec}(A) \times \mathbf{A}^n$. Suppose that V is the closure (with respect to the Zariski topology) of V° in $X \times \mathbf{A}^n$. Setting $y = (\mathfrak{m}, x_1, \dots, x_n) \in V^\circ$, it is easy to see that $\mathcal{O}_{x,X} \xrightarrow{\sim} \mathcal{O}_{y,V}$ and $p_2(y) = (0, \dots, 0)$. Since the composite map

$$V \hookrightarrow X \times \mathbf{A}^n \xrightarrow{p_2} \mathbf{A}^n$$

is proper and $\mathcal{O}_{(0), \mathbb{A}^n} \rightarrow \mathcal{O}_{y, V}$ is finite, we get $p_2^{-1}((0, \dots, 0)) \cap V = \{y\}$.

To prove (4) $\Rightarrow$ (1) in the case where X is a rational surface over $K = \overline{\mathbb{Q}}$, it is enough to show the following three lemmas.

Lemma 3.2. *Let $(A, \mathfrak{m})$ be a local domain essentially of finite type over a field K which is an algebraic extension (may be infinite) over the rationals $\mathbb{Q}$. Assume that there exists a Noether normalization D of A . Then A has a Noether normalization D' containing K .*

Lemma 3.3. *Let A be a localization at a maximal ideal of an affine domain over a field K of characteristic zero. Assume that A has a Noether normalization $(D, \mathfrak{n})$ containing K . Then D is essentially of finite type over K .*

In particular, D itself is a localization at a maximal ideal of an affine domain over K .

Lemma 3.4. *Let A be a local domain at a closed point of a rational surface over an algebraically closed field K of characteristic zero. Assume that A has a Noether normalization D essentially of finite type over K . Then D is isomorphic to $K[x, y]_{(x, y)}$ as a K -algebra.*

Proof of Lemma 3.2. Since D contains $\mathbb{Q}$, $L = Q(D) \cap K$ is contained in D . (This intersection is taken in $Q(A)$.) Let $Q(D) \cdot K$ be the smallest subfield of $Q(A)$ containing both $Q(D)$ and K . Since L is algebraically closed in $Q(D)$, the natural map $Q(D) \otimes_L K \rightarrow Q(D) \cdot K$ is the isomorphism. So we have $[K : L] = [Q(D) \cdot K : Q(D)] \leq [Q(A) : Q(D)] < \infty$. Then $D \otimes_L K$ is regular because K is étale over L . Since $D \otimes_L K$ is the subring of $Q(D) \cdot K$ generated by D and K , we get $D \subseteq D \otimes_L K \subset A$. Hence $D \otimes_L K$ is a Noether normalization of A containing K .

Proof of Lemma 3.3. Let A be a localization at a maximal ideal $\mathfrak{m}$ of an affine domain R and M the smallest Galois extension of $Q(D)$ containing $Q(R)$. Denote by G the Galois group $\text{Gal}(M/Q(D))$. Suppose C is the affine domain over K generated by $\{\sigma(R) \mid \sigma \in G\}$. (C is a subring of M .) Setting B as the normalization of C , G acts on B , i.e., $\sigma(B) = B$ for any $\sigma \in G$. Put $F = B^G = B \cap Q(D)$. Since B is an affine domain, so is F . Furthermore F is contained in D because $F = B \cap Q(D) \subseteq D' \cap Q(D) = D$, where D' is the integral closure of D in M . Put $\mathfrak{p} = F \cap \mathfrak{n}$. (It is easy to see that $\mathfrak{p}$ is a maximal ideal of F .)

We will prove $F_{\mathfrak{p}} = D$. Let $\{\mathfrak{m}_1, \dots, \mathfrak{m}_s\}$ be the set of maximal ideals of B lying over $\mathfrak{p}$.

First we show $B_{\mathfrak{m}_i} \cap Q(D) = F_{\mathfrak{p}}$ for every i . For $a \in B_{\mathfrak{m}_i} \cap Q(D)$, we define an ideal of $B_{(F \setminus \mathfrak{p})}$ as $I_a = \{b \in B_{(F \setminus \mathfrak{p})} \mid ab \in B_{(F \setminus \mathfrak{p})}\}$. Since $a \in B_{\mathfrak{m}_i}$, $I_a \not\subseteq \mathfrak{m}_i B_{(F \setminus \mathfrak{p})}$. It is easy to check that I_a is G -stable. So I_a is the unit ideal and a is contained in $B_{(F \setminus \mathfrak{p})}$. Hence we have $B_{\mathfrak{m}_i} \cap Q(D) = B_{(F \setminus \mathfrak{p})} \cap Q(D) = F_{\mathfrak{p}}$.

On the other hand, we can show $B_{\mathfrak{m}_i} \cap Q(D) \supseteq D$ for some i as follows. Denote by $R \cdot F$ the composite ring of R and F in A . Then $R \cdot F \hookrightarrow B$ is finite. Put $\mathfrak{q} = \mathfrak{m}A \cap R \cdot F$. Then we have $A = R_{\mathfrak{m}} \subseteq (R \cdot F)_{\mathfrak{q}} \subseteq A = R_{\mathfrak{m}}$. So $A = (R \cdot F)_{\mathfrak{q}}$. It is easy to see that $\mathfrak{q}$ is a maximal ideal of $R \cdot F$. Let $\mathfrak{m}_i$ be a maximal ideal of B such that $\mathfrak{m}_i \cap (R \cdot F) = \mathfrak{q}$. Then we get $B_{\mathfrak{m}_i} \supseteq A \supseteq D$. So we have $B_{\mathfrak{m}_i} \cap Q(D) \supseteq D$.

Therefore D coincides with $F_{\mathfrak{p}}$.

Proof of Lemma 3.4. It is easy to see that we may assume that D is a localization at a maximal ideal of a two-dimensional affine domain over K . Then $Q(D)$ is a purely transcendental extension over K by the Zariski-Castelnuovo theorem [4]. Let X be a two-dimensional smooth rational projective variety over K and x a closed point such that $\mathcal{O}_{X,x} = D$. Suppose $\pi: X \rightarrow X'$ is a birational morphism and X' is a relatively minimal model in the category of the rational surfaces over K . Put $\pi(x) = y$. It is known that $\mathcal{O}_{y,X'}$ is K -isomorphic to $K[x, y]_{(x,y)}$ (see [1, 2]). Since π can be factored into a finite sequence of blow-ups along one point, D itself is K -isomorphic to $K[x, y]_{(x,y)}$.

We have completed the proof of Theorem 3.1. Q.E.D.

4. SOME EXAMPLES

In this section we give some interesting examples. We will omit detailed calculations.

Example 4.1. Let $R = \mathbb{C}[x, y]/(f(x, y))$ and $f(x, y) = y^3 - (4x + 2)y^2 + (4x - x^2)y - (x^4 + 2x^3 + 2x^2)$. ($\mathbb{C}$ is the field of complex numbers.) Put $m = (x, y)$ and $A = R_m$. Then for any $h \in R$, the inclusion map $\mathbb{C}[h]_{(h)} \hookrightarrow A$ is not finite.

With notation as above,

$$R = \mathbb{C}[t^2 + t^3, t^2 + t^4] \subseteq \mathbb{C}[t]$$

and

$$A = \mathbb{C}[t^2 + t^3, t^2 + t^4]_{(t^2+t^3, t^2+t^4)} = \mathbb{C}[t^2, t^3]_{(t^2, t^3)} \subseteq \mathbb{C}[t]_{(t)}.$$

So we have a Noether normalization $\mathbb{C}[t^2]_{(t^2)} \hookrightarrow A$. But fixing an affine domain R and using the Noether normalization theorem for an affine domain R , we can never obtain a Noether normalization of A .

Example 4.2. Let K be an algebraic extension of the rationals $\mathbb{Q}$. Put $R = K[t^2 + t^3, t^6, t^7, t^8]_{(t^2+t^3, t^6, t^7, t^8)}$. Then for any Noether normalization D of R , $[Q(R) : Q(D)] \geq 3$ holds.

With notation as above we have $e(R) = 2$. It is easy to see that $t^2 + t^3$ is a minimal reduction of the maximal ideals of R and the completion $\hat{R} = K[[t^2 + t^3, t^6, t^7, t^8]]$. Then $K[[t^2 + t^3]] \hookrightarrow K[[t^2 + t^3, t^6, t^7, t^8]] = \hat{R}$ is finite and

$$[Q(\hat{R}) : Q(K[[t^2 + t^3]])] = \text{rank}_{K[[t^2+t^3]]} \hat{R} = e_{(t^2+t^3)R}(\hat{R}) = 2.$$

Hence $\hat{R}$ has a Noether normalization D such that $[Q(\hat{R}) : Q(D)] = e(\hat{R}) = 2$. But the above example implies that R does not have a Noether normalization D such that $[Q(R) : Q(D)] = e(R)$ although R has a Noether normalization by Theorem 2.3.

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