EVERY G -MODULE IS A SUBMODULE OF A DIRECT SUM OF CYCLICS

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ABSTRACT. Let G be a group and V a finite-dimensional complex G -module. It is shown that G is (isomorphic to) a submodule of a direct sum $W_1 \oplus \cdots \oplus W_s$ where each W_i is a cyclic finite-dimensional complex G -module. If G is an analytic (respectively algebraic) group and V is an analytic (respectively rational) module then the W_i can be taken to be analytic (respectively rational).

1. INTRODUCTION

Let G be a complex analytic group. As Singer has pointed out, [7, Corollary 2.15, p. 38] essentially asserts that every (finite-dimensional, analytic) G -module is a submodule of a direct sum of cyclic (finite-dimensional, analytic) G -modules. Unfortunately, the proof provided does not establish the assertion.

One can raise the same question—is every module a submodule of a direct sum of cyclic modules—for any group, and indeed for any ring. In this paper, attention will be restricted to modules finite-dimensional over an algebraically closed characteristic zero field k . It is easy to exhibit examples of rings for which the question has a negative answer (see Example 4.1 below). For groups, however, the answer turns out always to be positive; it is the purpose of this note to establish this result.

If V is a module for the group G , there is a natural homomorphism $G \rightarrow \mathrm{GL}(V)$. Let $\overline{G}$ be the Zariski closure of its image. Then V is also a rational $\overline{G}$ -module, while any rational $\overline{G}$ -module is by restriction a G -module. This observation reduces us to the case of linear algebraic groups. Since simple modules are cyclic, semisimple modules are direct sums of cyclic modules. So the key case to consider is that of a unipotent algebraic group. This is done in §2 below. In §3 the extension to the case of general algebraic groups, analytic groups, and groups in general is given. Section 4 is devoted to examples and remarks.

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2. UNIPOTENT GROUPS

Let U be a unipotent algebraic group over the (algebraically closed characteristic zero) field k . We recall some facts about the structure of the coordinate ring $k[U]$ as an algebra and as a U -module; the references are [6] and [8]. The notation to be introduced is to remain in force throughout this section.

First, we recall the structure of $k[U]$ as a U -module. U acts on $k[U]$ on the left by $(x \cdot f)(y) = f(yx)$ and on the right by $(f \cdot x)(y) = f(xy)$. The one-dimensional trivial U -module k lies in $k[U]$ as the constants, and $k[U]$, as a left U -module, is the injective hull of k in the category of rational U -modules [2]; we use $E(U)$ to denote $k[U]$ so regarded. We define an ascending chain of submodules $E_i(U)$ of $E(U)$ by $E_{-1}(U) = \{0\}$ and $E_{i+1}(U) = \{f \mid f \cdot x - f \in E_i(U) \forall x \in H\}$. When there is no confusion, we write E_i for $E_i(U)$.

The lower central series of U is given by $C^1 U = U$ and $C^{p+1} U = (U, C^p U)$. If $x_1, \dots, x_d$ in U are such that $\{x_i C^2 U\}$ is a k -basis of $C^1 U / C^2 U$, then $x_1, \dots, x_d$ generate U . Fix such a minimal generating set, and let X_p denote the endomorphism of $E(U)$ given by $X_i(f) = f \cdot x_i^{-1} - f$. Then

$$E_p(U) = \{f \in U \mid m \cdot f = 0 \forall m \text{ a monomial in the } X_i \text{ of weight } > p\}.$$

As a k -algebra, $k[U]$ is a polynomial ring. We choose any rational isomorphism $\text{Lie}(U) \rightarrow U$ and any k -basis of $\text{Lie}(U)$ to give polynomial generators. To link the algebra structure and the module structure, we make the choice as follows: let $\mathcal{E}^i \text{Lie}(U)$ denote the lower central series of $\text{Lie}(U)$, where $\mathcal{E}^1 \text{Lie}(U) = \text{Lie}(U)$ and $\mathcal{E}^{i+1} \text{Lie}(U) = [\text{Lie}(U), \mathcal{E}^i \text{Lie}(U)]$. Let elements $Y_{ij} \in \mathcal{E}^i \text{Lie}(U)$ be chosen so that $\{Y_{ij} + \mathcal{E}^i \text{Lie}(U)\}$ is a basis for $\mathcal{E}^i \text{Lie}(U) / \mathcal{E}^{i+1} \text{Lie}(U)$. Then $\{Y_{ij}\}$ is a basis of $\text{Lie}(U)$. Let $\{b_{ij}\}$ be the dual basis, and define $f_{ij} \in k[U]$ by $f_{ij} = b_{ij} \circ \log$. Then $k[U] = k[\{f_{ij}\}]$, and for each p , $E_p(U)$ has for a basis the set of all products

$$\prod f_{i_j}^{n_{i_j}} \quad \text{such that} \quad \sum i n_{i_j} \leq p$$

(see [6, Theorem 2.5, p. 54]). In other words, if we regard $k[U]$ as a graded polynomial ring where the generator f_{ij} has weight i , then E_p is spanned by the monomials of weight p or less.

Every U -endomorphism of $E(U)$ can be expressed as a formal power series in $X_1, \dots, X_d$ [5, Theorem 1.6, p. 327]. This applies in particular to the U invariant derivations of $k[U]$, namely to $\text{Lie}(U)$. We want to consider $E(U)$ as a module over the universal enveloping algebra $\mathcal{U}(\text{Lie}(U))$, and for our purposes it will be sufficient to consider its image $\mathcal{A}$ in $\text{End}_U(E(U))$. We have a natural filtration on $\text{End}_U(E(U))$ given by the degree of the leading monomial in a power series expansion of an endomorphism; that is, by the powers of the augmentation ideal of $k[X_1, \dots, X_d]$ (which by [5, Theorem 1.5, p. 327] consists of the endomorphisms vanishing on constants). This induces a filtration on $\mathcal{A}$. We isolate the main fact about this filtration in the following lemma.

Lemma 2.1. *Let $\mathcal{A}$ denote the subalgebra of $\text{End}_U(E(U))$ generated by $\text{Lie}(U)$, and let I denote the ideal of $\mathcal{A}$ consisting of all endomorphisms vanishing on k . If $T \in I^p$ and $f \in E_q(U)$, then $T \cdot f \in E_{q-p}(U)$.*

Proof. T can be expressed as a formal power series in $X_1, \dots, X_d$. Since $T \in I^p$, the monomials in T are of weight p or higher. Let m be such a monomial. If n is any monomial of weight $> q - p$, then $n \cdot m$ is of weight q or higher. Thus if $f \in E_q(U)$, we have $n \cdot m \cdot f = 0$. This holds for every monomial of weight $> q - p$, so we conclude that $m \cdot f \in E_q(U)$. It follows that $T \cdot f \in E_q(U)$.

We will want to apply Lemma 2.1 to expressions involving elements of $\text{Lie}(U)$. To that end, we recall that inside the universal enveloping algebra of a free Lie algebra (which is itself a free algebra), the degree of a Lie monomial in the Lie algebra is equal to its degree in the free algebra [9, Theorem 5.8, p. 323]. Since $\mathcal{E}^p \text{Lie}(U)$ is spanned by Lie monomials of weight p , this means the following in terms of the above filtration on $\mathcal{A}$.

Corollary 2.2. *In the notation of (1.1), if $D \in \mathcal{E}^p \text{Lie}(U)$, then $D \in I^p$.*

Next, we prove the main result of this section in the case that U has dimension one.

Lemma 2.3. *Assume $\dim(U) = 1$, and let f denote the additive character generating $k[U]$. Then $E_p(U)$ is cyclic, generated by f^p .*

Proof. Let x denote a generator of U . We will write x^a for $\exp(a \log x)$. Then f is given by $f(x^a) = a$, so $x \cdot f = f + 1$ and $f \cdot x^{-1} = f - 1$. Thus $x \cdot f^q = (f + 1)^q$, from which it follows that $\{g \mid \deg_f(g) \leq p\}$ is a U -stable submodule of $E(U)$ and also that if X denotes the transformation $X(g) = g \cdot x^{-1} - g$, then the $X^r(f^s)$ has maximal degree $s - r$ in f . Thus the set of elements of $E(U)$ annihilated by all X^r for $r > p$ is spanned by $\{f^i \mid i \leq p\}$, so E_p is that span. It is an exercise with the binomial theorem to pass from the formula $x \cdot f^q = (f + 1)^q$ (so that $q^{-1}(x \cdot f^q - f^q) = f^{q-1} + \text{lower degree}$) to the conclusion that f^p generates $E_p(U)$ as a U -module.

Now we come to our main result.

Theorem 2.4. *Let U be a unipotent algebraic group, and let $p \geq 0$. Then there is an element g in $E(U)$ such that the cyclic U -submodule generated by g contains $E_p(U)$.*

Proof. We will proceed by induction on the dimension of U ; the case of $\dim(U) = 1$ being covered by Lemma 2.3. We use the structure of $E(U)$ recalled above. Assume that $C^s U \neq 1$ and $C^{s+1} U = 1$, and let c denote the dimension of $C^s U = \mathcal{E}^s \text{Lie}(U)$. Above, we have selected a basis $Y_{s_1}, \dots, Y_{s_c}$ of $C^s U$ and corresponding functions $f_{s_1}, \dots, f_{s_c}$. For the rest of this proof, we will also denote Y_{s_c} by D and f_{s_c} by f . Let Z denote the subgroup $\exp(kD)$ of U ; Z is central because D is central in $\text{Lie}(U)$. We form the quotient $\bar{U} = U/Z$. Now $\dim(\bar{U}) = \dim(U) - 1$, so by induction we may assume that $E_p(\bar{U})$ is contained in the cyclic $\bar{U}$ -submodule of $E(\bar{U})$ generated by some element m , which we assume belongs to $E_r(\bar{U})$. As noted above, $E_p(U)$ is spanned by the monomials $\prod f_{i_j}^{n_{i_j}}$, where $\sum i n_{i_j} \leq p$. The surjection $U \rightarrow \bar{U}$ induces a surjection $\text{Lie}(U) \rightarrow \text{Lie}(\bar{U})$, which we will denote by overlining. By construction, the set $\{\bar{Y}_{ij} \mid 1 \leq i \leq s, 1 \leq j \leq c - 1\}$ forms a basis of $\text{Lie}(\bar{U})$ compatible with its lower central series (we make the obvious index adjustment if $c = 1$), and we can define the corresponding functions $\{\bar{f}_{ij}\}$ on

$\bar{U}$. The inclusion $k[\bar{U}] \rightarrow k[U]$ carries $\bar{f}_{ij}$ to the corresponding f_{ij} . We denote the image of $E_i(\bar{U})$ by $\bar{E}_i$. Note that $f \in E_s(U)$, but $f \notin E_{s-1}(U)$, that $\bar{E} = k[U]^2$, and that $E = \bar{E}[f] = \sum \bar{E}f^i$ (polynomial ring). With the above definitions of $\bar{E}_i$, the correspondence between the $\bar{f}_{ij}$ and their images in $E(U)$, and the given monomial basis, we have the formula

$$(*) \quad E_p(U) = \sum_{i=0}^{\ell} (\bar{E}_{p-is})f^i \quad \text{where } \ell = \sup \{j \mid sj \leq p\}.$$

Now let $g = mf^\ell$. We are going to show that the cyclic submodule generated by g contains E_p . We recall that $m \in \bar{E}_r$ and that $f \in E_s$. By (*) above, it suffices to prove that each $(\bar{E}_{p-is})f^i$ is contained in the submodule generated by g . As above, we let $\mathcal{A}$ be the subalgebra of $\text{End}_U(E(U))$ generated by $\text{Lie}(U)$. Its restriction $\bar{\mathcal{A}}$ to $\bar{E}$ is isomorphic to the subalgebra of $\text{End}_{\bar{U}}(E(\bar{U}))$ generated by $\text{Lie}(\bar{U})$. This means that $\bar{E}_p \subseteq \bar{\mathcal{A}} \cdot m$. In particular, if $y \in \bar{E}_{p-is}$, then there is a $T \in \bar{\mathcal{A}}$ such that $Tm = y$. Since $m \in \bar{E}_r$ and $y \in \bar{E}_{p-is}$, it follows from Lemma 2.1 that $T \in I^q$, where $q \geq r - (p - is)$. We want to consider the element $h = TD^{\ell-i}(mf^\ell)$. Since m lies in $\bar{E}_{\ell-1} = k[U]^2$ and D generates $\text{Lie}(Z)$, $D(m) = 0$. Also, $D(f^\ell) = \ell f^{\ell-1}$. So $D^{\ell-i}(mf^\ell)$ is a nonzero constant times mf^i and h is a constant multiple of $T(mf^i)$. We can write T as a sum of monomials in the derivations Y_{ij} where $i < s$ or $j < c$. Let Y be such a derivation. Then $Y(mf^i) = Y(m)f^i + imY(f)f^{i-1}$. Since $f \in E_s$ and $Y \in I$, $Y(f) \in E_{s-1}$, and of course no power of f can appear in E_{s-1} . Thus $Y(mf^i) = Y(m)f^i + \text{terms of lower degree in } f$, and by repeating this argument we discover that $T(mf^i) = T(m)f^i + \text{terms of lower degree in } f$; that is, h is a constant multiple of $yf^i + h'$, where the degree in f of h' is less than i . Since $T \in I^q$ for some $q \geq r - (p - is)$, $TD^{\ell-i}$ belongs to I^{r-p+ls} . Also, mf^ℓ belongs to E_{r+ls} , so h belongs to $E_{(r+ls)-(r-p+ls)} = E_p$. In other words, for each i , $0 \leq i \leq \ell$, and each $y \in \bar{E}_{p-is}$, the cyclic submodule M generated by y contains an element h of the form $yf^i + h'$ that belongs to $\bar{E}_p$ and has $\deg_f(h') < i$. Note that since $yf^i \in \bar{E}_p$, $h' = h - yf^i \in \bar{E}_p$. These statements also apply when p is replaced by $p - k$. Since $E_{p-k} = \sum_i (\bar{E}_{(p-k)-is})f^i$, we need only replace $\ell - i$ by the appropriate power e in forming $TD^e(mf^\ell)$, which is a constant multiple of $T(mf^i)$. In other words, for each element y of $\bar{E}_{p-k-is}$, there is an element in $M \cap \bar{E}_{p-k}$ of the form $yf^i + h'$ where the degree of h' in y is less than i .

Now we argue inductively on i that $\bar{E}_{p-is}f^i \subseteq M$ for $i = 0, 1, \dots, \ell$. For $i = 0$, this follows from the fact that $\bar{E}_p \subseteq \mathcal{A} \cdot m$. Suppose the inclusion held for all $j < i$, and let $yf^i \in \bar{E}_{p-i}f^i$. By the above, there is $h \in E_p \cap M$ of the form $h = yf^i + h'$ with $\deg_f(h') < i$. It follows that h' belongs to $\sum_{j=0}^{i-1} \bar{E}_{p-js}f^j$, where the sum ranges over j with $sj < i$. By the induction hypothesis, $h' \in M$. Thus $yf^i = h - h' \in M$, so $\bar{E}_{p-is}f^i \subseteq M$, and the proof of Theorem 2.4 is complete.

We observe that the induction step of Theorem 2.4 starts with an element in $E_r(\bar{U})$ generating a cyclic submodule containing $E_p(\bar{U})$ and producing an element in $E_{r\ell s}(U)$ generating a cyclic submodule containing $E_p(U)$, where ℓ is

$\sup\{j \mid sj \leq p\}$ and s is the length of the lower central series of U . This produces the following explicit form of Theorem 2.4:

Corollary 2.5. *Let $\{f_{ij}\}$ generate $k[U]$ as in the proof of (2.4). Fix $p \geq 0$, and let $\ell_i = \sup\{e \mid ie \leq p\}$. Then $E_p(U)$ is contained in the cyclic submodule of $E(U)$ generated by $\prod (f_{ij})^{\ell_i}$.*

Also, since $E(U)$ is the ascending union of its submodules $E_p(U)$, we can also restate Theorem 2.4 in the following form.

Corollary 2.6. *Let U be a unipotent group. Then any finitely generated submodule of $E(U)$ is contained in a cyclic submodule.*

Corollary 2.6 also remains true for pronipotent groups U , since any finitely generated submodule of $k[U]$ actually lies inside $k[\bar{U}]$ for some unipotent quotient $\bar{U}$ of U . (See [3–5] for the theory of pronipotent groups.) In fact, for the case of free pronipotent groups in the sense of [5, Definition 2.1, p. 83], we have a somewhat stronger statment (see Proposition 2.9 below). The following remarks about Lie algebras will play a role in that proof.

Proposition 2.7. *Let L be a (finite-dimensional) Lie algebra, $\mathcal{U}$ its universal enveloping algebra, and I the augmentation ideal of $\mathcal{U}$. Then*

- (1) *If $x \in I^n - I^{n+1}$ and $y \in I^m - I^{m+1}$, then $xy \in I^{n+m} - I^{n+m+1}$.*
- (2) *$\{\bar{x} \in \mathcal{U}/I^{n+1} \mid I \cdot \bar{x} = 0\} = I^n/I^{n+1}$.*

Proof. The associated graded ring $\text{gr}_I(\mathcal{U}) = \bigoplus I^p/I^{p+1}$ is canonically isomorphic to the universal enveloping algebra of $\text{gr}(L) = \bigoplus \mathcal{C}^p L/\mathcal{C}^{p+1} L$ [11] and hence is an integral domain. Thus (1) obtains. For (2), let $\bar{x} = x + I^{n+1}$ belong to the left-hand side. Assume $x \in I^m - I^{m+1}$ for $m = n - k$. If $k > 0$, choose $y \in I^k - I^{k+1}$. Then yx belongs to $I^{m+k} - I^{m+k+1} = I^n - I^{n+1}$, so $y\bar{x} \neq 0$. Thus $k = 0$, so $x \in I^n$, and we conclude that the left-hand side of (2) is contained on the right. The reverse inclusion is obvious so (2) is obtained.

This applies to unipotent groups as follows.

Lemma 2.8. *Let U be a unipotent group. Then*

$$(E_p(U)/E_{p-1}(U))^* = (E_p(U))^{*U};$$

that is, the socle of E_p^ is $(E_p/E_{p-1})^*$ and E_{p-1} is the Fratinni submodule of E_p .*

Proof. Let L denote $\text{Lie}(U)$, $\mathcal{U}$ denote the universal enveloping algebra of L , and I denote the augmentation ideal of $\mathcal{U}$. Then $E_p = (\mathcal{U}/I^{p+1})^*$ as an L -module [4, Example 1.8, p. 535], so $E_p^* = \mathcal{U}/I^{p+1}$ as an L -module. For any U -module M regarded as a $\mathcal{U}$ -module, we have $M^U = \{x \in M \mid Ix = 0\}$ so that Lemma 2.8 follows from the second part of Proposition 2.7.

Lemma 2.8 also applies to finitely generated pronipotent groups, since in that case all the modules involved factor through unipotent quotient groups.

For any unipotent (or pronipotent) group U , the module E_p/E_{p-1} has trivial U -action, as does the isomorphic (by Lemma 2.8) module $(E_p^*)^U$. This suggests trying to extend E_p^* by E_p by patching the socle of the former to the cosocle of the latter. As we shall now see, this can be done in the case of

the free pronipotent group, and the extension turns out to be a cyclic module containing E_p .

Proposition 2.9. *Let F be a free pronipotent group on d generators. Then there exists an F -submodule M of $E(F)$ containing E_p such that M/E_{p-1} is isomorphic to E_p^* . Moreover, M is cyclic.*

Proof. E/E_{p-1} is the direct sum of n copies of E , where $n = \dim((E/E_{p-1})^F)$ [5, Lemma 1.3.1, p. 326]. Moreover, $(E/E_{p-1})^F = E_p/E_{p-1}$ [5, Lemma 1.3.3, p. 326]. By Lemma 2.8, $(E_p^*)^F = (E_p/E_{p-1})^*$. We choose an isomorphism $(E_p/E_{p-1})^* \rightarrow E_p/E_{p-1}$, which then also embeds the source into the injective module E/E_{p-1} and hence extends to a homomorphism $h: E_p^* \rightarrow E/E_{p-1}$, which is a monomorphism since it is injective on the socle $(E_p^*)^F$ of E_p^* . Let M be the inverse image in E of $h(E_p^*)$. Then M contains E_{p-1} , and $M/E_{p-1} = E_p^*$. Let $\bar{M}$ be any semisimple quotient of M . (That is, quotient with trivial F -action.) Let $\bar{E}_p$ be the image of E_p in $\bar{M}$. Since this is a semi-simple quotient of E_p , Lemma 2.8 implies that E_{p-1} must have 0 image in $\bar{E}_p$, so $M/E_{p-1} = E_p^*$ maps onto $\bar{M}$. The socle of E_p is $E_0 = k$, so k is the cosocle of E_p^* . If the cosocle of a module is cyclic, so is the module (the kernel of the canonical projection onto the cosocle is the Fratinni submodule), and hence E_p^* is cyclic. But then its homomorphic image $\bar{M}$ is also cyclic. It then follows that the cosocle of M is cyclic, whence M itself is cyclic.

The referee has pointed out that when U is the unipotent radical of a Borel subgroup B of a semisimple, simply connected group G , then $k[U]$ is known to be the directed union of B -submodules that are obtained from irreducible G -modules (actually Steinberg modules) tensored with one-dimensional B -modules [1, 5.5, p. 107]. Since irreducible G -modules are U -cyclic (generated by highest weight vectors) and one-dimensional B -modules are U -trivial, it follows that $k[U]$ is the directed union of cyclic U -submodules. Actually, [1] deals with the case of positive characteristic, but, as the referee pointed out, essentially the same arguments apply in characteristic zero (and provide a rapid proof of Corollary 2.6 above in this special case). We want to record here the additional case of positive characteristic.

Proposition 2.10. *Assume that k is an algebraically closed field of positive characteristic. Let G be a connected, simply connected, semisimple algebraic group over k , and let U be the unipotent radical of a Borel subgroup of G . Then every finite dimensional U -submodule of $k[U]$ is contained in a cyclic submodule.*

3. GENERAL GROUPS

First we consider the case of linear algebraic groups.

We will need to recall the structure of injective rational modules over such groups G [2]: any such is a direct sum of indecomposable injective modules, which in turn have the form $E_G(V)$, the G -injective hull of the simple G -module V . If $G = U \cdot P$, where U is the unipotent radical of G and P is a maximal reductive subgroup, then $E(V) = k[G]^P \otimes V$.

Theorem 3.1. *Let G be a linear algebraic group over k , and let V be a finite-dimensional rational G -module. Then there are cyclic (finite-dimensional,*

rational) G -modules $W_1, \dots, W_d$ such that

$$V \subseteq \bigoplus_{i=1}^d W_i.$$

Proof. We consider the injective hull $E(V)$; this is a direct sum $E(V) = \bigoplus E(X_i)$, where $\bigoplus X_i$ is the socle of V . It suffices to prove that the projection of V to $E(X_i)$ is contained in a cyclic submodule, and hence we may assume that V has a simple socle X , so V is contained in $E_G(X)$.

Thus we need to prove that if X is a simple G -module and V is a finitely generated submodule of $E_G(X)$, then V is contained in a cyclic submodule. As above, we identify $E(X)$ with $k[G]^P \otimes X$, where P is a maximal reductive subgroup of G . Let U be the unipotent radical of G . As a left U -module, $k[G]^P$ is isomorphic to $k[U]$ via restriction of functions. As a U -module, $k[G]^P \otimes X = k[U]^{(n)}$ (direct sum), where $\dim(X) = n$. It follows that there is a p such that V is contained in $E_p(U) \otimes X$. We can also regard $E_p(U)$ as the p th term in the ascending socle series of $k[G]^P$, from which it follows that $E_p(U)$ is a G -submodule. Now choose, by Theorem 2.4, $h \in k[U]$ such that the cyclic U -submodule generated by h contains $E_p(U)$, and let $g \in k[G]^P$ restrict to h . Choose $x \in X$ nonzero, and let M be the cyclic G -submodule of $E_G(X)$ generated by $g \otimes x$. Since U acts trivially on X , M contains the span of $U(g \otimes x) = (Ug) \otimes x$, which in turn contains $E_p(U) \otimes x$. Thus M also contains $G(E_p(U) \otimes x) = E_p(U) \otimes Gx$ (recall that $GE_p(U) = E_p(U)$, since $E_p(U)$ is a G -module) and hence also its span $E_p(U) \otimes X$. So M also contains V , and then the theorem follows.

(The author is grateful to E. Cline for help with Theorem 3.1.)

We now extend Theorem 3.1 to analytic groups and abstract groups. These results are corollaries of Theorem 3.1 and some observations about Zariski density.

Corollary 3.2. *Let H be a complex analytic group and V a finite-dimensional analytic H -module. Then there are cyclic analytic H -modules $W_1, \dots, W_d$ such that $V \subseteq \bigoplus W_i$.*

Proof. Let G denote the Zariski closure of H in $\mathrm{GL}(V)$. We construct the cyclic G -modules $W_1, \dots, W_d$ and the embedding $V \rightarrow \bigoplus W_i$ by Theorem 3.1. The W_i are obviously also analytic H -modules by restriction. We claim that W_i is also H -cyclic; for suppose that w_i is a G -generator of W_i but that the span X of Hw_i is a proper subspace of W_i . Choose a nonzero linear functional ϕ on W_i vanishing on X . Then define f on G by $f(g) = \phi(gw_i)$. Then f is a rational function on G vanishing on the Zariski dense subgroup H , so f vanishes on G . This implies that ϕ is zero. This contradiction then implies that $X = W_i$ and hence that W_i is H -cyclic.

The only place that analyticity was used in Corollary 3.2 was to observe that rational G -modules are analytic H -modules by restriction. As a consequence, the proof for Corollary 3.2 also implies the following for the general case.

Corollary 3.3. *Let Γ be a group and V a finite-dimensional $k[\Gamma]$ -module. Then there are cyclic Γ -modules $W_1, \dots, W_d$ such that $V \subseteq \bigoplus W_i$.*

(The author is grateful to A. Lubotzky for pointing out the above corollaries.)

4. EXAMPLES

Our first example shows that there are (finite-dimensional, commutative) k -algebras that have modules finite-dimensional over k that are not embeddable in direct sums of cyclic modules (showing that Corollary 3.3 cannot be extended to rings).

Example 4.1. Let $R = k[x, y]/(x, y)^2 = k[\bar{x}, \bar{y}]$, and let $R^* = \text{Hom}_k(R, k)$. We will show that the R -module R^* cannot be embedded in a direct sum of cyclic R -modules.

First we determine the cyclic R -modules. Ideals of R correspond to ideals I of $k[x, y]$ containing $(x, y)^2$, and these are of the form $(x, y)^2 + (f)$, where $f = ax + by$ is linear and (x, y) . The corresponding cyclic modules R/I are then R , $k[(\bar{x}), \bar{y}]/(a\bar{x} + b\bar{y})$, and k , which have k dimension 3, 2, and 1 respectively. Now R^* has a basis $\hat{1}$, $\hat{x}$, and $\hat{y}$ dual to the basis $\bar{1}$, $\bar{x}$, $\bar{y}$ of R , and the R action is $\bar{x}\hat{x} = \hat{1}$, $\bar{y}\hat{y} = \hat{1}$, and the mixed products 0. It follows that R^* is an essential extension of its submodule $k\hat{1}$. Now suppose M is a cyclic R -module and $\phi: R^* \rightarrow M$ is an R -homomorphism. If M has dimension 3 and ϕ is injective then ϕ is an isomorphism, which is impossible (the socle of R has dimension 2, and that of R^* has dimension 1), so either M has dimension 2 or less, or ϕ is not injective. In either case, $\text{Ker}(\phi)$ is nonzero, so $\phi(\hat{1}) = 0$. Since every homomorphism from R^* to a cyclic module kills $\hat{1}$, there can be no embedding of R^* in a direct sum of cyclic R -modules.

Our next example illustrates Corollary 2.5 in the case of the Heisenberg group.

Example 4.2. The Heisenberg group U is the group of 3×3 upper triangular unipotent matrices. A typical element of U is the matrix

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

which we will denote by the triple (c, a, b) . Then $k[U]$ is the polynomial ring $k[x, y, z]$, where $x(c, a, b) = a$, $y(c, a, b) = b$, and $z(c, a, b) = c$. U acts on $k[U]$ by $g \cdot x = x + x(g)$, $g \cdot y = y + y(g)$, and $g \cdot z = z + z(g) + y(g)x$.

The Lie algebra of U can be identified with the Lie algebra L of strictly upper triangular matrices. L has a basis

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and X, Y give a basis of $\mathcal{E}^1 L / \mathcal{E}^2 L$, and Z is a basis of $\mathcal{E}^2 L$. Since

$$\exp(aX + bY + cZ) = I + aX + bY + cZ + \frac{1}{2}abZ,$$

we have, in the notation of §2 used in Corollary 2.5, $Y_{11} = X$, $Y_{12} = Y$, $Y_{21} = Z$, $f_{11} = x$, $f_{12} = y$, and $f_{21} = z - \frac{1}{2}xy$; for convenience we denote this last function by w . Then E_2 has basis $w, x^2, xy, y^2, x, y, 1$ (and we could replace w by z). By Corollary 2.5, E_2 is contained in the cyclic U -module generated by $h = x^2 y^2 w \in E_6$.

As we are now going to show, however, we can also exhibit an element of lower degree that also works.

When we interpret L as derivations of $k[U]$, we find that $X \leftrightarrow D_1 = -(\partial/\partial x + y\partial/\partial z)$, $Y \leftrightarrow D_2 = -\partial/\partial y$, and $Z \leftrightarrow D_3 = -\partial/\partial z$.

Let $q = x^2y^2 + z^2 \in E_4$. Then (up to constants) we have

$$\begin{aligned} D_1^2(q) &= y^2, \\ D_1D_2(q) &= xy, \\ D_2^2(q) &= x^2, \\ D_3(q) &= z, \end{aligned}$$

from which it follows easily that the cyclic U -module generated by q contains E_2 .

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