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# ON THE ALEKSANDROV PROBLEM OF CONSERVATIVE DISTANCES

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ABSTRACT. A case of the Aleksandrov problem for unit distance preserving mappings between metric spaces is solved. The relevance of methods used in mathematical foundations of quantum mechanics is shown for another case of Aleksandrov problem involving angular distances $\pi/2$ on the unit sphere.

## 1. INTRODUCTION

Let $X$ be a metric space with a metric $d: X \times X \to R^+ \cup \{0\}$. A. D. Aleksandrov has posed the following problem: Under what conditions is a mapping $f: X \to X$ preserving unit distance an isometry? (cf. [1]).

The problem has been solved for $X$ a finite-dimensional real Euclidean space $X = E^n$. If $n = 1$, then such a mapping $f$ does not need to be an isometry. If $2 \leq n < \infty$, $f$ must be an isometry due to the theorem of Beckman and Quarles [2]. For an infinite-dimensional Euclidean space $E^\infty$ the conclusion does not hold. An example of a unit distance preserving mapping that is not an isometry has been given by the second author [3]. All known examples of this kind involve discontinuous mappings. A problem, therefore, has been raised by the second author [3] whether a *continuous* mapping $f: E^\infty \to E^\infty$ preserving unit distance must be an isometry. In fact, a similar property in case of a sphere in a real or complex Hilbert space has been established by the first author [4]. We shall show below that the same is valid also if $f$ is a *homeomorphism* of $E^\infty$, thus giving a partial positive solution to a problem raised in [3].

A problem of similar nature has arised for $X$ being a unit sphere in $E^n$ $(1 \leq n \leq \infty)$ and $f: X \to X$ a mapping preserving the angular distance $\pi/2$ between the unit vectors (see [3]). Is $f$ necessarily an isometry? Here the answer turns out to be positive due to the theorems applied in the mathematical foundations of quantum mechanics.

## 2. CONSERVATIVE DISTANCES AND ISOMETRIES

Let $f: X \to X$ be any mapping of a metric space $X$ into itself. We shall call

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a nonnegative number $l$ a *conservative distance* of $f$ iff $d(x, y) = l$ implies $d(f(x), f(y)) = l$ for all $x, y$ in $X$.

Let $X = E^n$ $(1 \leq n \leq \infty)$. Generalizing slightly the problem of A. D. Aleksandrov and the problem raised in [3], we shall now consider a *homeomorphism* $f: X \to X$ having a conservative distance $l > 0$. One has the following

**Lemma 1.** *Let $f: E^n \to E^n$ $(1 \leq n \leq \infty)$ be a homeomorphism with a conservative distance $l > 0$. Then the surface of any sphere of radius $l$ is mapped homeomorphically onto the surface of another sphere of radius $l$.*

*Proof.* Indeed, since $l$ is $f$-conservative, any sphere $S(a, l) = \{x \in E^n: d(a, x) = l\}$ is transformed *into* the sphere $S(f(a), l)$. As $f$ is a homeomorphism of $E^n$, this mapping is continuous and different points of $S(a, l)$ have different images in $S(f(a), l)$. Moreover, the image $S^* = f(S(a, l)) = \{f(x): x \in S(a, l)\}$ must be exactly equal to $S(f(a), l)$. Indeed, let $Z$ be the complement of $S(a, l)$ and $Z^*$ the complement of $S^*$, i.e., $Z = E^n \backslash S(a, l)$ and $Z^* = E^n \backslash S^*$. Notice, that $Z$ is disconnected (it is the sum of two disjoint open sets, the *interior* and the *exterior* of the sphere $S(a, l)$. While mapping $S(a, l)$ onto $S^*$, the homeomorphism $f$ must simultaneously map $Z$ onto $Z^*$. Now, should $S^* \subset S(f(a), l)$ be a proper subset of $S(f(a), l)$, its complement $Z^*$ would be *connected*, which is impossible, since $Z^* = f(Z)$ and $Z$ is disconnected. Thus, $S^* = S(f(a), l)$. Q.E.D.

**Theorem 1.** *Every homeomorphism $f: E^n \to E^n$ $(3 \leq n \leq \infty)$ with a nontrivial conservative distance $l > 0$ is an isometry.*

*Proof.* Under our assumptions, the restriction of $f$ to any sphere $S(a, l)$, where $a \in E^n$, has a nontrivial conservative distance $l$. It corresponds to a nontrivial conservative angular distance $\alpha = \pi/3$ as the surface points $x, y \in S(a, l)$ with $d(x, y) = l$ and the center of the sphere form an equilateral triangle. If $n \geq 3$, the result of [4] implies that $f$ maps the sphere $S(a, l)$ *isometrically* onto $S(f(a), l)$. Since any two points $x, y \in E^n$ $(3 \leq n \leq \infty)$ with $d(x, y) \leq 2l$ lie on a surface of a certain sphere $S(a, l)$, this immediately proves that all distances $d \leq 2l$ are conservative. By iterating the argument, one sees that the distances $d \leq 2 \cdot 2l$ are conservative, too, and by induction it follows that every distance $d \geq 0$ is $f$-conservative. Q.E.D.

*Remark.* For any *finite* $n \geq 2$ the theorem follows as well from the result in [2]. The new element here is the validity of Theorem 1 when $f$ is a homeomorphism defined on an infinite-dimensional Euclidean space, thus solving a significant part of a problem raised in [3].

**Open Problems.** Some physical questions originated by optics or by quantum mechanics lead to the following concepts of *nonexpanding* or *nonshrinking* distances.

**Definition.** Let $f$ be a mapping of a metric space $X$ into itself. A nonnegative number $l$ is called a *nonexpanding distance* of $f$ iff $d(x, y) = l$ implies $d(f(x), f(y)) \leq l$ for every $x, y \in X$, and $l$ is called a *nonshrinking distance of $f$* iff $d(x, y) = l$ implies $d(f(x), f(y)) \geq l$ for every $x, y \in X$.

If $X$ is a sphere of a radius $r$ in a Euclidean space $E^n$ $(2 \leq n \leq \infty)$, the existence of a nontrivial nonexpanding (or nonshrinking) distance $l$ $(0 < l < 2r)$ for a homeomorphism $f: X \to X$ implies the existence of a conservative distance, thus proving that $f$ is an isometry [4]. For $X = E^n$ this implication does not hold. Indeed, for conformal nonisometric transformations defined on $E^n$ all distances are either nonshrinking or nonexpanding. An interesting problem would be to find the structure of a mapping $f: E^n \to E^n$ that admits simultaneously a nonshrinking distance $d_1$- and a nonexpanding distance $d_2$. A question arises whether such a mapping must be an isometry?

## 3. Euclidean spheres and the conservative angle $\pi/2$

For $X$ being a sphere in $E^n$ $(2 \leq n \leq \infty)$ a natural metric structure is defined on $X$ by the *angular distances.* The mappings that preserve the angular distance $\pi/2$ (i.e., orthogonality preserving) have been of particular interest for the mathematical foundations of quantum mechanics. What is their structure? Though the answer has been basically known for a long time, the last brick to the proof has been only recently added [5].

For convenience, we shall represent $X$ as the sphere of unit vectors in a real Hilbert space $H$. Now, for any subset $Z \subseteq X$ let $Z^\perp$ denote the set of all $x \in X$ orthogonal to all $z \in Z$. As it is easily seen, $Z \subseteq (Z^\perp)^\perp$.

The subsets $Z$ for which $Z = (Z^\perp)^\perp$ are of some special interest; they are intersections of the sphere $X$ with closed linear subspaces of $H$. For simplicity, we shall refer to them further on as *subspaces* of $X$. Two subspaces $Y$, $Z \subseteq X$ are called *orthogonal* (denoted by $Y \perp Z$) iff $y \perp z$ for every $y \in Y$ and $z \in Z$. The orthogonality of the subspaces of $X$ corresponds exactly to the orthogonality of the closed linear subspaces of $H$. For any two subspaces $Y$, $Z \subseteq X$ one defines the partial ordering relation "$\leq$" as the set theoretical inclusion, i.e., $Y \leq Z$ iff $Y \subseteq Z$. The set $L$ of all subspaces of $X$ with ordering relation "$\leq$" and with the mapping $Y \to Y^\perp$ is an orthocomplemented lattice isomorphic to the lattice of all closed linear subspaces of $H$. Now let $f: X \to X$ be a homeomorphism with $\pi/2$ being a conservative angle. As orthogonality preserving, $f$ induces a unique authomorphism of the orthocomplemented lattice $L$, which will be denoted by the same symbol $f$. Assume now that $3 \leq \dim H \leq +\infty$. Then, $f$ must conserve not only the orthogonality $(\perp)$ in $L$ but also the angles between the 1-dimensional subspaces (rays). This fact is a consequence of the existence of some more general numerical invariants on lattices and orthoposets. Indeed, consider the set $M$ of all $\sigma$-orthoadditive probability measures $\mu: L \to [0, 1]$. Assume that for any $X \in L$ there exists at least one measure $\mu \in M$ for which $\mu(X) = 1$ (this turns out to be true for our lattice of subspaces $L$). For any pair $X$, $Y \in L$ define now the quantity $p(X, Y)$ as the infimum on $Y$ of all probability measures $\mu$ taking the value 1 on $X$:

$$p(X, Y) = \inf_{\mu(X)=1} \mu(Y).$$

As an automorphism of $L$, $f$ induces an invertible transformation of the $\sigma$-orthoadditive measures $\mu \in M$ into new such measures $\mu \circ f^{-1}$, and since the value of $\mu$ on $X$ coincides with the value of $\mu \circ f^{-1}$ on $f(X)$, the infimum $p(X, Y)$ must stay invariant, i.e., $p(f(X), f(Y)) = p(X, Y)$. If now $X$ and $Y$ are two one-dimensional subspaces (rays) spanned by two unit vectors $x$, $y \in H$, then by the Gleason theorem [6] there is exactly one measure $\mu_X \in M$ with $\mu_X(X) = 1$, and the infimum $p(X, Y)$ becomes:

$$p(X, Y) = \mu_X(Y) = |\langle x, y \rangle|^2.$$

Henceforth, the quantity $|\langle x, y\rangle|^2$ must be invariant too, implying:

$$|\langle f(x), f(y)\rangle| = |\langle x, y\rangle|. \tag{3.1}$$

The following theorem can now be proved.

**Theorem 2.** *Let $f$ be a homeomorphism of the unit sphere $X$ in a real Hilbert space $H$ $(3 \leq \dim H \leq \infty)$ that conserves the angular distance $\pi/2$. Then $f$ is an isometry.*

*Proof.* Indeed, our mapping $f$ defines an authomorphism of the orthocomplemented lattice $L$, which for $3 \leq \dim H$, implies (3.1). Now, in virtue of Wigner theorem [7], which only recently has acquired a rigorous form [5], the conservation formula (3.1) implies that $f$ differs trivially from a certain linear isometry operator $I: X \to X$,

$$f(x) = k(x)Ix \tag{3.2}$$

where $I: X \to X$ is an isometry and $k(x)$ is just a real-valued function on $X$, $|k(x)| = 1$ for every $x$ in $X$. Since $I$ is continuous and $f$ is a homeomorphism, $k(x)$ must be continuous as well, which leaves only two possibilities, i.e., $k(x) \equiv +1$ or $k(x) \equiv -1$ everywhere on $X$. Thus, $f(x) \equiv Ix$ or $f(x) \equiv -Ix$ everywhere on $X$, meaning that $f$ is an isometry. Q.E.D.

*Remark.* Of course, Theorem 2, modulo aspects discussed by Sharma [5], has been known in the foundations of quantum theory, but we have considered it of some interest to bring methods of mathematical physics to understand better a part of the mathematical Aleksandrov problem.

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