ON WEIGHTED L^p -CONVERGENCE OF CERTAIN LAGRANGE INTERPOLATION

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ABSTRACT. The weighted L^p -convergence of Lagrange interpolation based on the zeros of $(1-x^2)T_n(x)T'_n(x)$ (where $T_n(x)$ is the Tchebycheff polynomial of the first kind) is considered. The corresponding result of A. Bellen (J. Approx. Theory 33 (1981), 85–95) is improved and perfected.

1. INTRODUCTION

Let

$$(1.1) \quad -1 \leq x_n < x_{n-1} < \cdots < x_1 \leq 1$$

be a matrix X of nodes. For a function f defined on $[-1, 1]$, let $L_n(f, x)$ be the corresponding Lagrange interpolation based on the nodes (1.1). If the matrix X is formed by the zeros of a Jacobi polynomial and $w(x) = w^{(\alpha, \beta)}(x) = (1-x)^\alpha(1+x)^\beta$ is a Jacobi weight function, then a classical theorem of Erdős and Turán [3] states

$$(1.2) \quad \lim_{n \rightarrow \infty} \int_{-1}^1 w(x) [L_n(f, w, x) - f(x)]^2 dx = 0$$

for $f \in C[-1, 1]$, where $L_n(f, w, x)$ is the Lagrange interpolation based on the zeros of a Jacobi polynomial. As was proved by Askey [4, 5] the power 2 generally cannot be replaced by $2 + \varepsilon$, but corresponding to the Tchebycheff weight $w^{(-1/2, -1/2)}(x) = (1-x^2)^{-1/2}$, Erdős and Feldheim [6] showed

Theorem A. *Let $f \in C[-1, 1]$. Then*

$$(1.3) \quad \lim_{n \rightarrow \infty} \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} |L_n(f, w^{(-1/2, -1/2)}, x) - f(x)|^p dx = 0$$

for any fixed $p > 0$.

In [1, 2] Bellen raised the following problem:

Find a matrix Y of additional nodes:

$$(1.4) \quad -1 \leq y_{m(n)} < y_{m(n)-1} < \cdots < y_1 \leq 1, \quad x_i \neq y_j \quad \forall i, j,$$

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where $m(n) = an + b$, a and b are real, and $a > 0$, such that for the new matrix

$$(1.5) \quad X \cup Y \quad x_n, \dots, x_1; y_{m(n)}, \dots, y_1$$

the Lagrange interpolation polynomial $L_{n+m(n)}(f, x)$ based on the nodes (1.4) satisfy

$$(1.6) \quad \lim_{n \rightarrow \infty} \int_{-1}^1 w(x) [L_{n+m(n)}(f, x) - f(x)]^2 dx = 0$$

for some class of function f defined on $[-1, 1]$.

If the matrix X is formed by the zeros of $(1 - x^2)U_{n-1}(x)$ (where $U_{n-1}(x)$ is the Tchebycheff polynomial of the second kind) and the additional matrix Y is formed by the zeros of $T_n(x)$ (where $T_n(x)$ is the Tchebycheff polynomial of the first kind), Bellen [2] proved that for the corresponding operator $L_{2n+1}(f, x)$ we have

Theorem B. *Let $f \in \text{Lip } \gamma$, $\gamma > \frac{1}{2}$. Then*

$$(1.7) \quad \lim_{n \rightarrow \infty} \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} [L_{2n+1}(f, x) - f(x)]^2 dx = 0.$$

Theorem B is a special case ($\alpha = -\frac{1}{2}$) of a more general theorem in [2] concerning Gegenbaues polynomials $p_n^{(\alpha)}(x)$, $-1 < \alpha < 0$. Indeed, for the zeros of $(1 - x^2)U_{n-1}(x)T_n(x)$, (1.7) was proved by Erdős and Turán for any $f \in C[-1, 1]$.

The purpose of this article is to introduce a method that gives an essential improvement of Theorem B. We can obtain the Erdős-Feldheim type theorem.

Theorem 1. *Let $f \in C[-1, 1]$. Then we have*

$$(1.8) \quad \lim_{n \rightarrow \infty} \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} |L_{2n+1}(f, x) - f(x)|^p dx = 0 \quad (p > 0).$$

It is obvious that Theorem B is improved and perfected.

2. PRELIMINARIES

Now we state and prove several lemmas that will be needed later on.

Lemma 1 [7]. *Let $f \in C[-1, 1]$. Then there exists a polynomial $p_n(x)$ of degree $\leq n$ such that*

$$(2.1) \quad |f(x) - P_n(x)| \leq c\omega\left(f, \frac{\sqrt{1-x^2}}{n}\right)$$

where $\omega(f, \cdot)$ is the modulus of continuity of f and c denotes an absolute positive constant that is independent of n .

Let

$$(2.2) \quad -1 = x_n < x_{n-1} < \dots < x_1 < x_0 = 1$$

be the zeros of $(1 - x^2)U_{n-1}(x)$ and

$$(2.3) \quad -1 < y_n < y_{n-1} < \dots < y_1 < 1$$

be the zeros of $T_n(x)$. Then Lagrange interpolation based on the zeros of $(1-x^2)T_n(x)T'_n(x)$ is

$$\begin{aligned} L_{2n+1}(f, x) &= \sum_{k=0}^n f(x_k) \frac{T_n(x)}{T_n(x_k)} l_k^*(x) + \sum_{k=1}^n f(y_k) \frac{(1-x^2)U_{n-1}(x)}{(1-y_k^2)U_{n-1}(y_k)} l_k(x) \\ (2.4) \quad &= T_n(x) \sum_{k=0}^n (-1)^k f(x_k) l_k^*(x) \\ &\quad + \sqrt{1-x^2} U_{n-1}(x) \sum_{k=1}^n (-1)^{k-1} f(y_k) \lambda_k(x) \end{aligned}$$

where

$$(2.5) \quad l_k^*(x) = \frac{(1-x^2)U_{n-1}(x)}{[(1-x^2)U_{n-1}(x)]'_{x=x_k}(x-x_k)}, \quad k = 0, 1, \dots, n,$$

$$(2.6) \quad \lambda_k(x) = \frac{\sqrt{1-x^2}}{\sqrt{1-y_k^2}} l_k(x) = \frac{\sqrt{1-x^2}}{\sqrt{1-y_k^2}} \frac{T_n(x)}{T'_n(y_k)(x-y_k)}, \quad k = 1, \dots, n.$$

Lemma 2. For every integer $m \geq 3$ there exists a constant c_m such that

$$(2.7) \quad \sum_{k=1}^n \lambda_k^2(x) \leq 5, \quad |\lambda_k(x)| \leq \sqrt{5},$$

and

$$(2.8) \quad \sum_{k=1}^n |\lambda_k^m(x)| \leq c_m.$$

Proof. One notes that

$$(2.9) \quad 1-x^2 = 1-y_k^2 - 2y_k(x-y_k) - (x-y_k)^2$$

and

$$(2.10) \quad \sum_{k=1}^n \frac{1}{x-y_k} = \frac{T'_n(x)}{T_n(x)}.$$

From (2.10) and (2.9) we have

$$\begin{aligned} \sum_{k=1}^n \lambda_k^2(x) &= \sum_{k=1}^n \frac{1-x^2}{1-y_k^2} l_k^2(x) \\ (2.11) \quad &= \sum_{k=1}^n \frac{1-y_k^2 - 2y_k(x-y_k) - (x-y_k)^2}{1-y_k^2} l_k^2(x) \\ &= \sum_{k=1}^n l_k^2(x) - 2 \sum_{k=1}^n \frac{y_k}{1-y_k^2} (x-y_k) l_k^2(x) - \sum_{k=1}^n \frac{(x-y_k)^2}{1-y_k^2} l_k^2(x) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Using Fejér's result (see [9]) we get

$$(2.12) \quad I_1 \leq 2.$$

From (2.10) we have

$$(2.13) \quad \begin{aligned} I_2 &= -\frac{2T_n^2(x)}{n^2} \sum_{k=1}^n \frac{y_k}{x-y_k} = \frac{2T_n^2(x)}{n^2} \sum_{k=1}^n \left(1 - \frac{x}{x-y_k}\right) \\ &= \frac{2T_n^2(x)}{n} - \frac{2xT_n(x)U_{n-1}(x)}{n}; \end{aligned}$$

therefore, we have

$$(2.14) \quad I_3 = -\frac{T_n^2(x)}{n}.$$

One notes that

$$(2.15) \quad |U_{n-1}(x)| \leq n-1,$$

$$(2.16) \quad |T_n(x)| \leq 1;$$

therefore combining (2.11)–(2.14) we obtain (2.7). Finally, for every $m \geq 3$ (2.8) follows from (2.7). This proves Lemma 2.

Lemma 3. Let $\gamma_1, \gamma_2, \dots, \gamma_k$ be distinct integers between 1 and n . Then we have ($k = 2m$)

$$(2.17) \quad \left| \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \lambda_{\gamma_1}(x) \lambda_{\gamma_2}(x) \cdots \lambda_{\gamma_k}(x) dx \right| \leq \frac{c}{n^{2m}}.$$

Proof. One notes that

$$(2.18) \quad \cos^{2m-1} n\theta = \frac{1}{2^{2(m-1)}} \sum_{j=0}^{m-1} \binom{2m-1}{j} \cos(2m-2j-1)n\theta$$

and

$$(2.19) \quad \sin^{2m} \theta = \frac{1}{2^{2m-1}} \left[\sum_{j=0}^{m-1} (-1)^{m+j} \binom{2m}{j} \cos(2m-2j)\theta + \frac{1}{2} \binom{2m}{m} \right].$$

From (2.18) and (2.19) we can obtain

$$(2.20) \quad (1-x^2)^m T_n^{2m-1}(x) = a_{n-2m} T_{n-2m}(x) + \sum_{j>n-2m}^{(2m-1)(n+1)+1} \mu_j \cos j\theta$$

where

$$(2.21) \quad a_{n-2m} = \frac{(-1)^m}{2^{4m-3}} \cdot \frac{(2m-1)!}{m!(m-1)!}.$$

It is also easy to see that

$$(2.22) \quad \frac{T_n(x)}{(x-y_{\gamma_1}) \cdots (x-y_{\gamma_k})} = 2^{2m} T_{n-2m}(x) + q_{n-2m-1}(x)$$

where $q_{n-2m-1}(x)$ is a polynomial of degree $\leq n-2m-1$. Using the Stirling formula

$$(2.23) \quad n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n (1 + \varepsilon_n) \quad \left(\lim_{n \rightarrow \infty} \varepsilon_n = 0\right),$$

together with the orthogonality property of the Tchebycheff polynomials, and from (2.6) and (2.20)–(2.23) we can obtain

$$\left| \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \lambda \gamma_1(x) \lambda \gamma_2(x) \cdots \lambda \gamma_k(x) dx \right| = \frac{\pi}{2^{2m-4}} \cdot \frac{(2m-1)!}{m!(m-1)!} \cdot \frac{1}{n^{2m}} \leq \frac{c}{n^{2m}}.$$

Therefore, we obtain (2.17). This proves Lemma 3.

3. PROOF OF THEOREM 1

Proof. Using (2.1) and the uniqueness of Lagrange interpolation, we have

$$\begin{aligned} L_{2n+1}(f, x) - f(x) &= L_{2n+1}(f - P_n, x) + P_n(x) - f(x) \\ &= T_n(x) \sum_{k=0}^n (-1)^k \Delta_n(x_k) l_k^*(x) \\ &\quad + \sqrt{1-x^2} U_{n-1}(x) \sum_{k=1}^n \Delta_n(y_k) \bar{\lambda}_k(x) + \Delta_n(x) \\ &= A_n(x) + B_n(x) + \Delta_n(x) \end{aligned} \quad (3.1)$$

where $\bar{\lambda}_k(x) = (-1)^{k-1} \lambda_k(x)$, $\Delta_n(x) = P_n(x) - f(x)$. Therefore, we have (for any fixed m)

$$\begin{aligned} &\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} |L_{2n+1}(f, x) - f(x)|^{2m} dx \\ &\leq c \left[\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} A_n^{2m}(x) dx + \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} B_n^{2m}(x) dx \right. \\ &\quad \left. + \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \Delta_n^{2m}(x) dx \right] \\ &= c(I_{1n} + I_{2n} + I_{3n}). \end{aligned} \quad (3.2)$$

Due to (2.1) we have

$$\lim_{n \rightarrow \infty} I_{3n} = 0. \quad (3.3)$$

Using [8, Theorem 2.1] and (2.16) we have

$$\lim_{n \rightarrow \infty} I_{1n} = 0. \quad (3.4)$$

Next, we note that

$$\sqrt{1-x^2} |U_n(x)| \leq 1. \quad (3.5)$$

From (3.2) and (3.5) we can obtain

$$I_{2n} \leq I_n := \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \left[\sum_{k=1}^n \Delta_n(y_k) \bar{\lambda}_k(x) \right]^{2m} dx.$$

One notes that

$$\begin{aligned} I_n &= \sum_{r_1+r_2+\cdots+r_s=2m} \frac{(2m)!}{r_1! r_2! \cdots r_s!} \Delta_n^{r_1}(y_{i_1}) \Delta_n^{r_2}(y_{i_2}) \cdots \Delta_n^{r_s}(y_{i_s}) \\ &\quad \cdot \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \bar{\lambda}_{i_1}^{r_1}(x) \bar{\lambda}_{i_2}^{r_2}(x) \cdots \bar{\lambda}_{i_s}^{r_s}(x) dx \end{aligned} \quad (3.6)$$

where $\sum$ denotes multiple sum and $0 \leq r_j \leq 2m$, $j = 1, 2, \dots, s$. We note that the first term (single sum) in (3.6) is

$$(3.7) \quad I_{n_1} = \sum_{k=1}^n \Delta_n^{2m}(y_k) \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \bar{\lambda}_k^{2m}(x) dx.$$

By using Lemmas 1 and 2 we can obtain

$$(3.8) \quad \lim_{n \rightarrow \infty} I_{n_1} = 0$$

since the second term (double sum) in (3.6) is

$$(3.9) \quad \begin{aligned} I_{n_2} &= \sum_{r=1}^{2m-1} \sum_{k \neq j} \sum_{j=1}^n \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} (\Delta_n(y_k) \bar{\lambda}_k(x))^r (\Delta_n(y_j) \bar{\lambda}_j(x))^{2m-r} dx \\ &= \sum_{r=1}^{2m-1} I_{n_2(r)}. \end{aligned}$$

For $r = 1$, we have

$$I_{n_2(1)} = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \sum_{k=1}^n \Delta_n(y_k) \bar{\lambda}_k(x) \sum_{j=1}^n (\Delta_n(y_j) \bar{\lambda}_j(x))^{2m-1} dx - I_{n_1}.$$

Using Cauchy's inequality and Lemmas 1-3, for $m > 2$ we have

$$(3.10) \quad \begin{aligned} & \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \left| \sum_{k=1}^n \Delta_n(y_k) \bar{\lambda}_k(x) \right| dx \\ & \leq \sqrt{\pi} \left(\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \left[\sum_{k=1}^n \Delta_n(y_k) \bar{\lambda}_k(x) \right]^2 dx \right)^{1/2} \\ & \leq c \left[\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \sum_{k=1}^n \Delta_n^2(y_k) \bar{\lambda}_k^2(x) dx \right. \\ & \quad \left. + \sum_{k \neq j} |\Delta_n(y_k) \Delta_n(y_j)| \cdot \left| \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \bar{\lambda}_k(x) \bar{\lambda}_j(x) dx \right| \right]^{1/2} \\ & \leq c\omega \left(f, \frac{1}{n} \right) \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

From (3.8), (3.10), and Lemma 2 we have

$$(3.11) \quad \lim_{n \rightarrow \infty} I_{n_2(1)} = 0.$$

Similarly, for $r = 2m - 1$ we have

$$(3.12) \quad \lim_{n \rightarrow \infty} I_{n_2(2m-1)} = 0.$$

If $2 \leq r \leq 2m - 2$, using Lemmas 1 and 2 we have

$$(3.13) \quad \lim_{n \rightarrow \infty} I_{n_2(r)} = 0 \quad (2 \leq r \leq 2m - 2).$$

Combining (3.9) and (3.11)–(3.13), we can obtain

$$(3.14) \quad \lim_{n \rightarrow \infty} I_{n_2} = 0.$$

Using similar methods of estimation of (3.14), we can obtain also

$$(3.15) \quad \lim_{n \rightarrow \infty} I_{n_i} = 0 \quad (3 \leq i \leq 2m - 1)$$

where I_{n_i} is the i th term (i -fold sum) in I_n . For the $2m$ th term ($2m$ -fold sum) in I_n , we have

$$I_{n_{2m}} = \sum_{k \neq j \neq \dots \neq i} \overbrace{\sum \dots \sum}^{2m} \Delta_n(y_k) \Delta_n(y_j) \dots \Delta_n(y_i) \\ \cdot \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \bar{\lambda}_k(x) \bar{\lambda}_j(x) \dots \bar{\lambda}_i(x) dx.$$

From Lemma 1 and 3 we can obtain $\lim_{n \rightarrow \infty} I_{n_{2m}} = 0$. Combining (3.8) and (3.14)–(3.16), we can obtain (1.8) for any $p = 2m$ and hence for any integer p . This proves Theorem 1.

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