THE CESÀRO OPERATOR IS BOUNDED ON H^p FOR $0 < p < 1$

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ABSTRACT. In this note the boundedness of the Cesàro operator on H^p , $0 < p < 1$, is proved.

1. INTRODUCTION

For $0 < p < \infty$, H^p denotes the standard Hardy space on the unit disc. The Cesàro operator on H^p is

$$\mathcal{C}(f)(z) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^n a_k \right) z^n$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is in H^p . It has been known that $\mathcal{C}$ is bounded on H^p for $1 \leq p < \infty$. Good results about the norm and the spectrum of $\mathcal{C}$ on H^p , $1 \leq p < \infty$, were obtained by using a semigroup in [3]. Recently Siskakis [4] gave a short proof of the boundedness of $\mathcal{C}$ on H^1 . It is natural to ask whether $\mathcal{C}$ is bounded on H^p for $0 < p < 1$. The present note will give a positive answer.

As usual, $M_p(r, f)$ denotes the integral mean on $|z| = r$ of an analytic f . Let $\|f\|_p = \sup_{0 \leq r < 1} M_p(r, f)$. In order to prove our result, we will make use of the following results, which we state as three lemmas.

Lemma 1 [1, p. 12]. *If $0 < p < \infty$ and $f \in H^p$, then there is a constant A depending only on p such that*

$$\int_0^{2\pi} \sup_{0 \leq r < 1} |f(re^{i\theta})|^p d\theta \leq A \|f\|_p^p.$$

Lemma 2 [1, p. 65]. *If $s > 1$ and $0 \leq r < 1$, then there is a constant B depending only on s such that*

$$\int_0^{2\pi} |1 - re^{i\theta}|^{-s} d\theta \leq B(1-r)^{-s+1}.$$

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Lemma 3 [2, p. 412]. If $0 < p < \infty$, $n > 1$, and $f \in H^p$, then there is a constant C depending only on p and n such that

$$\int_0^1 (M_{np}(r, f))^p (1-r)^{-1/n} dr \leq C \|f\|_p^p.$$

The above lemmas are due to Hardy and Littlewood. Lemma 1 is the well-known Hardy-Littlewood maximal theorem. Our ideas originate from [2].

2. THE RESULT AND THE PROOF

Theorem. If $0 < p < 1$ and $f \in H^p$, then there is a constant K depending only on p such that $\|\mathcal{E}(f)\|_p \leq K \|f\|_p$.

Proof. A calculation with power series gives

$$\mathcal{E}(f)(z) = \frac{1}{z} \int_0^z f(\zeta) \frac{1}{1-\zeta} d\zeta = \int_0^1 f(tz) \frac{1}{1-tz} dt.$$

For $0 \leq r < 1$, $0 < p < 1$, and $f \in H^p$, we have

$$\begin{aligned} (M_p(r, \mathcal{E}(f)))^p &= \frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^1 f(tre^{i\theta}) \frac{1}{1-tre^{i\theta}} dt \right|^p d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^1 \left| \frac{f(tre^{i\theta})}{1-tre^{i\theta}} \right| dt \right)^p d\theta. \end{aligned}$$

Let $t_k = 1 - 2^{-k}$ and $F_k(re^{i\theta}) = \sup_{t_{k-1} < t < t_k} |f(tre^{i\theta})/(1-tre^{i\theta})|$. Then

$$\int_0^1 \left| \frac{f(tre^{i\theta})}{1-tre^{i\theta}} \right| dt = \sum_{k=1}^{\infty} \int_{t_{k-1}}^{t_k} \left| \frac{f(tre^{i\theta})}{1-tre^{i\theta}} \right| dt \leq \sum_{k=1}^{\infty} F_k(re^{i\theta}) 2^{-k}.$$

Since $0 < p < 1$, we get

$$\left(\int_0^1 \left| \frac{f(tre^{i\theta})}{1-tre^{i\theta}} \right| dt \right)^p \leq \sum_{k=1}^{\infty} F_k^p(re^{i\theta}) 2^{-pk}$$

by the lemma of [1, p. 57]. Then Lemma 1 and the monotonicity of the integral mean give

$$\begin{aligned} (M_p(r, \mathcal{E}(f)))^p &\leq \frac{1}{2\pi} \sum_{k=1}^{\infty} 2^{-pk} \int_0^{2\pi} F_k^p(re^{i\theta}) d\theta \\ &\leq \frac{1}{2\pi} \sum_{k=1}^{\infty} 2^{-pk} \int_0^{2\pi} \sup_{0 \leq t < t_k} \left| \frac{f(tre^{i\theta})}{1-tre^{i\theta}} \right|^p d\theta \\ &\leq \frac{A}{2\pi} \sum_{k=1}^{\infty} 2^{-pk} \left(M_p \left(t_k r, \frac{f}{1-z} \right) \right)^p \\ &\leq D \sum_{k=1}^{\infty} \int_{t_k}^{t_{k+1}} (1-t)^{p-1} \left(M_p \left(tr, \frac{f}{1-z} \right) \right)^p dt \\ &\leq D \int_0^1 (1-t)^{p-1} \left(M_p \left(t, \frac{f}{1-z} \right) \right)^p dt \end{aligned}$$

where $D = Ap2^p/2\pi(2^p - 1)$. We can choose n such that $1 - p < 1/n < 1$. Let $1/m + 1/n = 1$. Then $mp = np/(n - 1) > 1$. Now Hölder's inequality and Lemma 2 give

$$\begin{aligned} \left(M_p\left(t, \frac{f}{1-z}\right)\right)^p &= \frac{1}{2\pi} \int_0^{2\pi} \left|\frac{f(te^{i\theta})}{1-te^{i\theta}}\right|^p d\theta \\ &\leq \frac{1}{2\pi} \left(\int_0^{2\pi} |f(te^{i\theta})|^{np} d\theta\right)^{1/n} \left(\int_0^{2\pi} |1-te^{i\theta}|^{-mp} d\theta\right)^{1/m} \\ &\leq \frac{B^{1/m}}{2\pi} (1-t)^{-p+1/m} \left(\int_0^{2\pi} |f(te^{i\theta})|^{np} d\theta\right)^{1/n} \\ &= (2\pi)^{1/n-1} B^{1/m} (1-t)^{-p+1-1/n} (M_{np}(t, f))^p. \end{aligned}$$

Combining the above inequalities, we get

$$\begin{aligned} (M_p(r, \mathcal{C}(f)))^p &\leq (2\pi)^{1/n-1} DB^{1/m} \int_0^1 (M_{np}(t, f))^p (1-t)^{-1/n} dt \\ &\leq K^p \|f\|_p^p \end{aligned}$$

by applying Lemma 3 in the last step. This completes the proof.

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