PRESCRIBING CURVATURE WITH NEGATIVE TOTAL CURVATURE ON OPEN RIEMANN SURFACES

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ABSTRACT. On open Riemann surfaces, we obtain a necessary and sufficient condition for conformal metrics to have prescribed curvature and the desired singularities leading to negative total curvature.

0. INTRODUCTION

Suppose M is a compact Riemann surface with Euler characteristic $\chi(M)$. Let g be a compatible metric with its Gaussian curvature K . The problem of prescribed curvature is as follows: for a given function $\tilde{K} \in C^\infty(M)$, is there a metric $\tilde{g}$ that is conformal to g and has $\tilde{K}$ as its curvature (cf. [KW])? If we write $\tilde{g} = e^{2u}g$, this problem is equivalent to solving the elliptic equation

$$(1) \quad \Delta u + \tilde{K}e^{2u} = K$$

on M , where Δ is the Laplacian under the metric g . If we integrate (1) over M we find that the total curvature of $\tilde{g}$ agrees with that of g and is fixed by the Gauss-Bonnet Theorem to be $2\pi\chi(M)$.

Let Γ be a finite set $\{p_1, \dots, p_n\}$ of points on M , and let $\widehat{M} = M \setminus \Gamma$. Then $\widehat{M}$ is an open surface. The problem of prescribed curvature on $\widehat{M}$ can be stated as follows: for a given function $\tilde{K} \in C(\widehat{M})$, does there exist a metric $\tilde{g}$ on $\widehat{M}$ that is conformal to g and has $\tilde{K}$ as its curvature on $\widehat{M}$? In this case, we should find a solution of the equation (1) on $\widehat{M}$. Since $\widehat{M}$ is noncompact the total curvature of $\tilde{g}$ need not agree with that of g ; the difference depends on the asymptotic behaviour of the solution u near the points p_i . In this paper, the asymptotic behaviour of the solution u contributes singularities to $\tilde{g}$.

As a special case we may take $\widehat{M} = \mathbf{R}^2 = S^2 \setminus \{p\}$. Several authors have considered the prescribed curvature problem on $\mathbf{R}^2$ (cf. [N, M2, M3, A, CN]). A good survey is due to Kazdan (cf. [K]). Recently, many of the results on $\mathbf{R}^2$ have been extended to the more general open surfaces $\widehat{M}$ by Hulin and Troyanov [HT] and McOwen [M1].

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For example, Ni [N] and McOwen [M3] considered the problem of prescribed curvature with negative total curvature on $\mathbf{R}^2$. They found that if $\tilde{K} \leq 0$ and $\tilde{K} = O(|x|^{-l})$ for $l > 2$, then there are conformal metrics with $\tilde{K}$ as its curvature and negative total curvature on $\mathbf{R}^2$. This result was extended to the more general surfaces $\widehat{M}$ in [HT, M1].

As another example, in [M2] McOwen used the variational method to obtain results on positive total curvature on $\mathbf{R}^2$; but, in addition, he observed that a necessary and sufficient condition for a conformal metric on $\mathbf{R}^2$ to have curvature $\tilde{K} = O(|x|^{-l})$ for $l > 2$ and zero total curvature is:

- (i) $\tilde{K}$ changes sign on $\mathbf{R}^2$,
- (ii) $\int \tilde{K} dx < 0$.

These results were extended to the more general surfaces $\widehat{M}$ in [HT]; in particular, the necessary and sufficient conditions for $\tilde{K}$ to be the curvature of a conformal metric with zero total curvature become

- (i*) $\tilde{K}$ changes sign on $\widehat{M}$,
- (ii*) $\int \tilde{K} dA_{g_{\bar{\alpha}}} < 0$,

where $g_{\bar{\alpha}}$ is a flat conformal metric representing the divisor $\bar{\alpha}$ on M (cf. §1, Definition and Proposition).

In this paper, we shall study the prescribed curvature problem with negative total curvature, but removing the assumption $\tilde{K} \leq 0$ that exists in all previous works (cf. [N, M1, M3, HT]). We shall find that the condition (ii*) is as decisive for negative total curvature as it was for zero total curvature: if (ii*) holds under some flat conformal metric $g_{\bar{\alpha}}$ representing the divisor $\bar{\alpha}$ on M , then there are conformal metrics with $\tilde{K}$ as their curvature and negative total curvature; in other words, condition (ii*) implies that the equation (1) on $\widehat{M}$ has solutions with the prescribed asymptotic behaviour near the points p_i . Moreover, condition (ii*) is also necessary for the existence of solutions of (1) with the desired asymptotic behaviour near the points p_i (cf. Theorems A, B). In the special case $\mathbf{R}^2 = S^2 \setminus \{p\}$, we obtain condition (ii*) as a necessary and sufficient condition on $\mathbf{R}^2$ (cf. Corollary). This result was not obtained in the earlier work on $\mathbf{R}^2$.

1. MAIN THEOREMS

Throughout this paper, without loss of generality, we assume that the metric g is Euclidean in a neighborhood of each point p_i (cf. [M1]). Denote $r_i = \text{dist}(x, p_i)$ under the metric g . Following Hulin and Troyanov (cf. [HT]), we shall use the following terminology.

Definition. On a compact Riemann surface M , a metric g_{α} is said to *represent the divisor* $\alpha := \sum_{i=1}^n \alpha_i p_i$, $\alpha_i \in \mathbf{R}$, if there is $\nu \in C^{\infty}(\widehat{M}) \cap H_2^p(M)$ for some $p > 1$ such that $g_{\alpha} = e^{2\nu} r_i^{2\alpha_i} g$ in a neighborhood of each point p_i . In this case, the *Euler characteristic* $\chi(M, \alpha)$ of (M, α) is defined by

$$\chi(M, \alpha) = \chi(M) + \sum_{i=1}^n \alpha_i.$$

Remark. By $\nu \in C^{\infty}(\widehat{M}) \cap H_2^p(M)$ we mean that $\nu \in H_2^p(M)$ and is C^{∞} on

$\widehat{M}$. Also, the more general conditions $\nu \in L^1(M)$ and $\Delta\nu \in L^1(M)$ (weakly) that are used in [HT] will not be necessary in our case.

We first observe that if $\chi(M, \bar{\alpha}) = 0$ then we can represent $\bar{\alpha}$ by a flat conformal metric. This was observed by Hulin and Troyanov (cf. [HT, Theorem 7.1]), but we shall give a proof in §2 for the sake of completeness.

Proposition. *Let M be a compact Riemann surface, let $\Gamma = \{p_1, \dots, p_n\} \subset M$, and let $\widehat{M} = M \setminus \Gamma$. If $\bar{\alpha} := \sum_{i=1}^n \bar{\alpha}_i p_i$ is a divisor with $\chi(M, \bar{\alpha}) = 0$, then there exists a flat conformal metric $g_{\bar{\alpha}} = e^{2u} g$ representing $\bar{\alpha}$. Moreover, $g_{\bar{\alpha}}$ is unique up to homothety (multiplication of $g_{\bar{\alpha}}$ by a positive constant).*

Next, we investigate the role $g_{\bar{\alpha}}$ plays in representing a divisor α by a conformal metric $g_{\alpha} = e^{2u} g$. The following result provides a necessary condition. For the sake of convenience, we define for any two divisors $\alpha := \sum_{i=1}^n \alpha_i p_i$ and $\beta := \sum_{i=1}^n \beta_i p_i$ having same support Γ ,

- (a) $|\alpha - \beta| = \max_{1 \leq i \leq n} |\alpha_i - \beta_i|$;
- (b) $\alpha \leq \beta$, if $\alpha_i \leq \beta_i$ for each i .

Theorem A. *Suppose M is a compact Riemann surface. Let $\Gamma = \{p_1, \dots, p_n\} \subset M$ and $\widehat{M} = M \setminus \Gamma$. Suppose $\tilde{K} \in C^\infty(\widehat{M})$ satisfies $\tilde{K} = O(r_i^{l_i})$ as $r_i \rightarrow 0$ and $\sum_{i=1}^n l_i > 2(\chi(M) - n) = 2\chi(\widehat{M})$. If there exists a conformal metric g_{α} representing a divisor $\alpha := \sum_{i=1}^n \alpha_i p_i$ with $\tilde{K}$ as its curvature, if $\chi(M, \alpha) < 0$, and if $\alpha_i > (l_i + 2)/2$ for each i , then for any divisor $\bar{\alpha} := \sum_{i=1}^n \bar{\alpha}_i p_i$ with $\chi(M, \bar{\alpha}) = 0$ and $\alpha \leq \bar{\alpha}$, we have $\int_M \tilde{K} dA_{g_{\bar{\alpha}}} < 0$, where $g_{\bar{\alpha}}$ is a flat conformal metric representing $\bar{\alpha}$ and $dA_{g_{\bar{\alpha}}}$ is the area element under the metric $g_{\bar{\alpha}}$.*

Finally, we show that the condition $\int_M \tilde{K} dA_{g_{\bar{\alpha}}} < 0$ is sufficient to represent divisors α near $\bar{\alpha}$ by conformal metrics g_{α} having curvature $\tilde{K}$.

Theorem B. *Suppose M is a compact Riemann surface. Let $\Gamma = \{p_1, \dots, p_n\} \subset M$ and $\widehat{M} = M \setminus \Gamma$. Suppose $\tilde{K} \in C^\infty(\widehat{M})$ satisfies $\tilde{K} = O(r_i^{l_i})$ as $r_i \rightarrow 0$ and $\sum_{i=1}^n l_i > 2(\chi(M) - n) = 2\chi(\widehat{M})$. If $\int_M \tilde{K} dA_{g_{\bar{\alpha}}} < 0$ for a flat conformal metric $g_{\bar{\alpha}}$ representing a divisor $\bar{\alpha} := \sum_{i=1}^n \bar{\alpha}_i p_i$ with $\chi(M, \bar{\alpha}) = 0$, and if $\bar{\alpha}_i > -(l_i + 2)/2$ for each i , then there is $\varepsilon > 0$ such that for any divisor $\alpha := \sum_{i=1}^n \alpha_i p_i$ satisfying $\chi(M, \alpha) < 0$ and for $|\alpha - \bar{\alpha}| < \varepsilon$, there exists a conformal metric g_{α} representing the divisor α with $\tilde{K}$ as its curvature.*

Remark. In Theorems A and B, there is a restriction on the l_i 's implied by $\chi(M, \alpha) < 0$ and $\alpha_i > -(l_i + 2)/2$ for each i ; namely, $\sum_{i=1}^n l_i > 2(\chi(M) - n)$. The total curvature $\tilde{k}$ under the conformal metric g_{α} is

$$\tilde{k} = \int_M \tilde{K} e^{2u} dA_0 = \int_M (K - \Delta u) dA_0 = 2\pi\chi(M, \alpha)$$

where dA_0 is the area element under the metric g . Thus, $\tilde{k} < 0$. By Proposition, condition (ii*) in Theorems A and B does not depend on which flat conformal metric we choose in the homothety class.

If we apply Theorems A and B to the case $\widehat{M} = S^2 \setminus \{p\}$, then we obtain the following necessary and sufficient condition for solving the prescribed curvature problem on $\mathbf{R}^2$.

Corollary. Suppose $\tilde{K} \in C^2(\mathbb{R}^2)$ satisfies the condition that $\tilde{K} = O(|x|^{-l})$ as $|x| \rightarrow \infty$ for some $l > 2$. Then there exists $\varepsilon > 0$ such that for any λ , $0 < \lambda < \varepsilon$, the equation

$$\Delta u + \tilde{K}e^{2u} = 0 \quad \text{in } \mathbb{R}^2$$

has a solution u with the asymptotic behaviour

$$u = \lambda \log |x| + c + o(1) \quad \text{as } |x| \rightarrow \infty$$

where c is a constant if and only if $\int_{\mathbb{R}^2} \tilde{K} dx < 0$.

2. PROOFS OF THE THEOREMS

In this section, we first give a proof of Proposition that is slightly different from the one in [HT].

Proof of Proposition. Choose $V_{\tilde{\alpha}} \in C^\infty(\widehat{M})$ with $V_{\tilde{\alpha}} = \tilde{\alpha}_i \log r_i + \nu$ in a neighborhood of each point p_i , where $\nu \in C^\infty(\widehat{M}) \cap H_2^p(M)$ for some $p > 1$. Note that $K - \Delta V_{\tilde{\alpha}}$ can be extended in $L^p(M) \cap C^\infty(\widehat{M})$ and

$$\int_M (K - \Delta V_{\tilde{\alpha}}) dA_0 = 2\pi \left(\chi(M) + \sum_{i=1}^n \tilde{\alpha}_i \right) = 0$$

where dA_0 is the area element under the metric g . Thus there exists $u_0 \in H_2^p(M) \cap C^\infty(\widehat{M})$ such that $\Delta u_0 = K - \Delta V_{\tilde{\alpha}}$. If $\nu \in C^\infty(M)$ then $u_0 \in C^\infty(M)$. Let $u_1 = u_0 + V_{\tilde{\alpha}}$. So $g_{\tilde{\alpha}} = e^{2u_1} g$ is a flat conformal metric representing $\tilde{\alpha}$. For any flat conformal metric $e^{2w_1} g$ representing $\tilde{\alpha}$, we have $u_1 - w_1 \in H_2^p(M) \cap C^\infty(\widehat{M}) \subset C(M) \cap C^\infty(\widehat{M})$. Since $\Delta(u_1 - w_1) = 0$, so $u_1 - w_1$ is a constant.

Then we prove our necessary condition in order for the function $\tilde{K}$ to be the curvature of a conformal metric representing the divisor α .

Proof of Theorem A. Let $g_\alpha = e^{2u} g$ be a conformal metric representing α with $\tilde{K}$ as its curvature, and let $g_{\tilde{\alpha}} = e^{2u_1} g$ be a flat conformal metric representing $\tilde{\alpha}$ with $\chi(M, \tilde{\alpha}) = 0$ and $\alpha \leq \tilde{\alpha}$. Thus

$$\begin{aligned} \Delta u + \tilde{K}e^{2u} &= K, \\ \Delta u_1 &= K, \\ \tilde{K}e^{2u_1} &= -\Delta(u - u_1)e^{-2(u-u_1)} \quad \text{on } \widehat{M}. \end{aligned}$$

By the assumption, $\int_M \tilde{K}e^{2u_1} dA_0$ is finite and

$$\int_M \tilde{K}e^{2u_1} dA_0 = - \int_M \Delta(u - u_1)e^{-2(u-u_1)} dA_0.$$

Let $B_\varepsilon(p_i) = \{x \in M : \text{dist}(x, p_i) < \varepsilon\}$ and $B_\varepsilon(\Gamma) = \bigcup_{i=1}^n B_\varepsilon(p_i)$. By the

Green's formula,

$$\begin{aligned} & \int_M \Delta(u - u_1) e^{-2(u - u_1)} dA_0 \\ &= \lim_{\varepsilon \rightarrow 0} \int_{M \setminus B_\varepsilon(\Gamma)} \Delta(u - u_1) e^{-2(u - u_1)} dA_0 \\ &= \lim_{\varepsilon \rightarrow 0} \left[\sum_{i=1}^n \int_{\partial B_\varepsilon(p_i)} \frac{\partial(u - u_1)}{\partial n_i} e^{-2(u - u_1)} ds_i \right. \\ & \quad \left. - \int_{M \setminus B_\varepsilon(\Gamma)} \nabla(u - u_1) \nabla e^{-2(u - u_1)} dA_0 \right] \end{aligned}$$

where n_i is the unit normal towards p_i on $\partial B_\varepsilon(p_i)$ and ds_i is the length element under the metric on $\partial B_\varepsilon(p_i)$ induced by the metric g . If $\alpha_i < \bar{\alpha}_i$, as $\varepsilon \rightarrow 0$

$$\left| \int_{\partial B_\varepsilon(p_i)} \frac{\partial(u - u_1)}{\partial n_i} e^{-2(u - u_1)} ds_i \right| \leq M_i \varepsilon^{2(\bar{\alpha}_i - \alpha_i)} \rightarrow 0$$

for some $M_i > 0$. If $\alpha_i = \bar{\alpha}_i$, as $\varepsilon \rightarrow 0$

$$\int_{\partial B_\varepsilon(p_i)} \frac{\partial(u - u_1)}{\partial n_i} e^{-2(u - u_1)} ds_i = o(1) \rightarrow 0.$$

Hence

$$\int_M \Delta(u - u_1) e^{-2(u - u_1)} dA_0 = 2 \int_M |\nabla(u - u_1)|^2 e^{-2(u - u_1)} dA_0 > 0,$$

which implies $\int_M \tilde{K} e^{2u_1} dA_0 < 0$. Therefore, $\int_M \tilde{K} dA_{g_\alpha} < 0$.

In order to prove the existence of the solution u of equation (1) with the desired asymptotic behaviour near the points p_i , we first rewrite equation (1), then construct upper and lower solutions. Suppose $V_i \in C^\infty(M \setminus \{p_i\})$ satisfies $V_i = \log r_i$ in a neighborhood of p_i . Let $V_\alpha = \sum_{i=1}^n \alpha_i V_i$ for a divisor $\alpha := \sum_{i=1}^n \alpha_i p_i$. Then $V_\alpha \in C^\infty(\widehat{M})$ satisfies $V_\alpha = \alpha_i \log r_i + V_0$ in a neighborhood of each point p_i , where $V_0 \in C^\infty(M)$. Let $V_{\bar{\alpha}} = \sum_{i=1}^n \bar{\alpha}_i V_i$ for a divisor $\bar{\alpha} := \sum_{i=1}^n \bar{\alpha}_i p_i$ with $\chi(M, \bar{\alpha}) = 0$. By the proof of Proposition, there is a unique $u_0 \in C^\infty(M)$ up to an additive constant such that $\Delta u_0 = K - \Delta V_{\bar{\alpha}}$ on $\widehat{M}$. Since

$$K - \Delta V_\alpha - \frac{2\pi\chi(M, \alpha)}{\text{area}(M)} \in C^\infty(M) \quad \text{and} \quad \int_M \left[K - \Delta V_\alpha - \frac{2\pi\chi(M, \alpha)}{\text{area}(M)} \right] dA_0 = 0,$$

there is a unique $U_\alpha \in C^\infty(M)$ up to an additive constant such that

$$\Delta U_\alpha = K - \Delta V_\alpha - 2\pi\chi(M, \alpha)/\text{area}(M).$$

We may choose $\int_M u_0 dA_0 = 0$, and $\int_M U_\alpha dA_0 = 0$. Let $u = W + U_\alpha + V_\alpha$ on $\widehat{M}$. Then equation (1) can be transformed into

$$(2) \quad \Delta W + \tilde{K}_\alpha e^{2W} = c_\alpha$$

on $\widehat{M}$, where $\tilde{K}_\alpha = \tilde{K} e^{2(U_\alpha + V_\alpha)}$ and $c_\alpha = 2\pi\chi(M, \alpha)/\text{area}(M)$. In order to solve equation (1) for u with $u = \alpha_i \log r_i + \nu$ in a neighborhood of each point

p_i , where $\nu \in C^\infty(\widehat{M}) \cap H_2^p(M)$ for some $p > 1$, we ought to solve equation (2) for $W \in C^\infty(\widehat{M}) \cap H_2^p(M)$. In [KW, §10], Kazdan and Warner discussed the equation $\Delta u = c - he^u$ with $c < 0$ in the $C^\infty(M)$ category. Here in equation (2), $\tilde{K}_\alpha \in L^p(M)$ for some $p > 1$. So we shall modify the approach in [KW] and find a solution of equation (2) in $C^\infty(\widehat{M}) \cap H_2^p(M)$. To prove Theorem B, we introduce the following lemmas.

Lemma 1. Suppose $\tilde{K} \in C^\infty(\widehat{M})$ is not identically zero, satisfies $\tilde{K} = O(r_i^{l_i})$ as $r_i \rightarrow 0$, and $\sum_{i=1}^n l_i > 2(\chi(M) - n)$. Let $W^+ \in C^\infty(\widehat{M})$ be a bounded upper solution of equation (2), i.e.,

$$\Delta W^+ + \tilde{K}_\alpha e^{2W^+} \leq c_\alpha \quad \text{on } \widehat{M}.$$

If the divisor $\alpha := \sum_{i=1}^n \alpha_i p_i$ satisfies $\chi(M, \alpha) < 0$ and $\alpha_i > -(l_i + 2)/2$ for each i , then there exists a bounded lower solution $W^- \in C^\infty(\widehat{M})$, i.e.,

$$\Delta W^- + \tilde{K}_\alpha e^{2W^-} \geq c_\alpha$$

and $W^- \leq W^+$ on $\widehat{M}$. In this case, equation (2) has a bounded solution $W \in C^\infty(\widehat{M}) \cap H_2^p(M)$ for some $p > 1$.

Proof. Let $h_\alpha = |\tilde{K}_\alpha|$. Then $\int_M h_\alpha dA_0$ is finite and positive. Let

$$\tilde{f}_\alpha = c_\alpha - 2\pi\chi(M, \alpha)h_\alpha / \int_M h_\alpha dA_0.$$

Since $\alpha_i > -(l_i + 2)/2$ for each i , so $\tilde{f}_\alpha \in L^p(M)$ for some $p > 1$. Note that $\int_M \tilde{f}_\alpha dA_0 = 0$. Thus there exists $W_1 \in H_2^p(M) \subset C^\beta(M)$ for $0 < \beta < 2 - 2/p$ such that $\Delta W_1 = \tilde{f}_\alpha$ on M . Hence $W_1 \in C^\infty(\widehat{M}) \cap C(M)$. Let $W^- = W_1 - \lambda$ for some positive constant λ . We have

$$\begin{aligned} \Delta W^- + \tilde{K}_\alpha e^{2W^-} &= c_\alpha - \frac{2\pi\chi(M, \alpha)h_\alpha}{\int_M h_\alpha dA_0} + \tilde{K}_\alpha e^{2W_1 - 2\lambda} \\ &\geq c_\alpha - h_\alpha \left[\frac{2\pi\chi(M, \alpha)}{\int_M h_\alpha dA_0} + e^{2W_1 - 2\lambda} \right] \geq c_\alpha \end{aligned}$$

and $W^- \leq W^+$ if a sufficiently large λ is chosen. By the process in [M4] and [M1], it is easy to see the existence of a solution $W \in C^\infty(\widehat{M}) \cap H_2^p(M)$ to equation (2).

Lemma 2. Let $U \in H_2^p(M)$ with $\int_M U dA_0 = 0$ for $p > 1$. Then

$$\|U\|_{C^\beta} \leq d \|\Delta U\|_p,$$

where $0 < \beta < 2 - 2/p$ and d is a constant independent of U .

Proof. This follows from the Sobolev's imbedding theorem and the standard elliptic estimates.

Lemma 3. Suppose $\tilde{K} \in C^\infty(\widehat{M})$ satisfies $\tilde{K} = O(r_i^{l_i})$ as $r_i \rightarrow 0$ and $\sum_{i=1}^n l_i > 2(\chi(M) - n)$. Let the divisor $\bar{\alpha} := \sum_{i=1}^n \bar{\alpha}_i p_i$ satisfy $\chi(M, \bar{\alpha}) = 0$ and $\bar{\alpha}_i > -(l_i + 2)/2$ for each i . For any divisor $\alpha := \sum_{i=1}^n \alpha_i p_i$ satisfying $\alpha_i > -(l_i + 2)/2$ for each i , let $h_\alpha = |\tilde{K}_\alpha|$ and $C_\alpha = \int_M \tilde{K}_\alpha dA_0$. Then we have

$$C_\alpha \rightarrow \int_M \tilde{K} dA_{g_\alpha}$$

and

$$\int_M h_\alpha dA_0 \rightarrow \int_M |\tilde{K}| dA_{g_{\bar{\alpha}}} \quad \text{as } \alpha \rightarrow \bar{\alpha}$$

where $g_{\bar{\alpha}}$ is a flat conformal metric representing $\bar{\alpha}$.

Proof. Claim that $U_\alpha \rightarrow u_0$ as $\alpha \rightarrow \bar{\alpha}$. Notice that

$$\Delta(U_\alpha - u_0) = f_\alpha \quad \text{with} \quad \int_M (U_\alpha - u_0) dA_0 = 0,$$

where $f_\alpha = -\Delta(V_\alpha - V_{\bar{\alpha}}) - c_\alpha$, and $f_\alpha \rightarrow 0$ as $\alpha \rightarrow \bar{\alpha}$. By Lemma 2,

$$|U_\alpha - u_0|_{C^\beta} \leq d \|f_\alpha\|_2 \quad \text{for } 0 < \beta < 1.$$

Hence, $U_\alpha \rightarrow u_0$ as $\alpha \rightarrow \bar{\alpha}$. Thus, $\tilde{K}_\alpha = \tilde{K}e^{2(U_\alpha + V_\alpha)}$ converges to $\tilde{K}e^{2(u_0 + V_\alpha)}$ pointwise on $\widehat{M}$ as $\alpha \rightarrow \bar{\alpha}$. By Dominated Convergence Theorem, we have

$$C_\alpha \rightarrow \int_M \tilde{K} dA_{g_{\bar{\alpha}}}$$

and

$$\int_M h_\alpha dA_0 \rightarrow \int_M |\tilde{K}| dA_{g_{\bar{\alpha}}} \quad \text{as } \alpha \rightarrow \bar{\alpha},$$

where $g_{\bar{\alpha}} = e^{2(u_0 + V_{\bar{\alpha}})} g$. From the proof of Proposition, we know that $g_{\bar{\alpha}}$ is a flat conformal metric representing $\bar{\alpha}$.

Recall that we want to solve equation (2). Using the condition $\int \tilde{K} dA_{g_{\bar{\alpha}}} < 0$ and Lemma 3, we shall construct a bounded upper solution W^+ of equation (2). From Lemma 1, it follows that equation (2) admits a bounded solution, thereby proving Theorem B.

Proof of Theorem B. Let $a_\alpha = \sqrt{|\alpha - \bar{\alpha}|}$, and observe

$$|\chi(M, \alpha)| \leq \sum_{i=1}^n |\alpha_i - \bar{\alpha}_i| \leq na_\alpha^2, \quad |c_\alpha| \leq \frac{2\pi na_\alpha^2}{\text{area}(M)}.$$

Now set

$$\hat{f}_\alpha = \left[C_\alpha - \frac{2\pi\chi(M, \alpha)}{a_\alpha} \right] \frac{h_\alpha}{\int_M h_\alpha dA_0} + \frac{c_\alpha}{a_\alpha} - \tilde{K}_\alpha.$$

Note that $\hat{f}_\alpha \in L^p(M)$ for some $p > 1$ and $\int_M \hat{f}_\alpha dA_0 = 0$. Then there exists $W_\alpha \in H_2^p(M)$ such that $\Delta W_\alpha = \hat{f}_\alpha$ with $\int_M W_\alpha dA_0 = 0$. But $H_2^p(M) \subset C^\beta(M)$ for $0 < \beta < 2 - 2/p$, together with local regularity imply $W_\alpha \in C^\infty(\widehat{M}) \cap C(M)$. From Lemma 2, $|W_\alpha|_{C^\beta} \leq d \|\hat{f}_\alpha\|_p$ for $0 < \beta < 2 - 2/p$. By Lemma 3, $\|\hat{f}_\alpha\|_p$ is bounded as $\alpha \rightarrow \bar{\alpha}$. Thus W_α is bounded as $\alpha \rightarrow \bar{\alpha}$. Let $W^+ = a_\alpha W_\alpha + b_\alpha$, and $e^{2b_\alpha} = a_\alpha$. From Lemma 3, we have

$$C_\alpha \rightarrow \int_M \tilde{K} dA_{g_{\bar{\alpha}}} < 0$$

and

$$\int_M h_\alpha dA_0 \rightarrow \int_M |\tilde{K}| dA_{g_{\bar{\alpha}}} > 0 \quad \text{as } \alpha \rightarrow \bar{\alpha}.$$

Then

$$\begin{aligned}\Delta W^+ + \tilde{K}_\alpha e^{2W^+} - c_\alpha &= \frac{[a_\alpha C_\alpha - 2\pi\chi(M, \alpha)]h_\alpha}{\int_M h_\alpha dA_0} + a_\alpha \tilde{K}_\alpha (e^{2a_\alpha W_\alpha} - 1) \\ &\leq \frac{a_\alpha h_\alpha}{\int_M h_\alpha dA_0} \left[C_\alpha + 2\pi n a_\alpha + |e^{2a_\alpha W_\alpha} - 1| \int_M h_\alpha dA_0 \right] \leq 0,\end{aligned}$$

as $\alpha \rightarrow \bar{\alpha}$, which proves that W^+ is a bounded upper solution. From Lemma 1, it follows that there exists a solution $W \in H_2^p(M) \cap C^\infty(\bar{M})$ of equation (2).

Proof of Corollary. Note that under the stereographic projection $\mathbf{P}: \mathbf{R}^2 \rightarrow S^2 \setminus \{p\}$, $4|x|^{-1} \approx \text{dist}(\mathbf{P}x, p)$ as $|x| \rightarrow \infty$ and the Euclidean metric on $\mathbf{R}^2$ can induce a flat conformal metric on S^2 representing a divisor $\bar{\alpha} := -2p$ with $\chi(S^2, \bar{\alpha}) = \chi(S^2) - 2 = 0$. Thus, under this projection, by translating Theorems A and B on S^2 into $\mathbf{R}^2$, we have this corollary.

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