NONEXISTENCE OF MEASURABLE OPTIMAL SELECTIONS

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ABSTRACT. We give an example of a function f on a separable metric space X into a compact metric space Y such that the graph of f is a Borel subset of $X \times Y$, but f is not Borel measurable. The example forms the basis for our construction of an upper semicontinuous, compact model of a one-day dynamic programming problem where the player has an optimal action at each state, but is unable to make a choice of such an action in a Borel measurable manner.

1. INTRODUCTION

A very useful result for dynamic programming/gambling/discrete time stochastic control is the following selection theorem due to Schäl [8, 9]:

Theorem A. *Suppose $(T, \mathcal{T})$ is a measurable space, and let Y be a separable metric space. Suppose G is a $\mathcal{T}$ -measurable multifunction on T such that $G(t)$ is a nonempty, compact subset of Y for every $t \in T$. Let v be a real-valued function on $\text{Graph}(G) (= \{(t, y) \in T \times Y : y \in G(t)\})$ satisfying the following condition:*

(1) *v is the pointwise limit of a nonincreasing sequence $\{v_n\}$ of real-valued functions on $\text{Graph}(G)$ such that each v_n is $(\mathcal{T} \times \mathbf{B}_Y | \text{Graph}(G))$ -measurable and $v_n(t, \cdot)$ is continuous on $G(t)$ for every $t \in T$, where $\mathbf{B}_Y$ is the Borel σ -field on Y . Let*

$$v^*(t) = \sup_{y \in G(t)} v(t, y), \quad t \in T.$$

Then there is a $\mathcal{T}$ -measurable selector $g: T \rightarrow Y$ of G such that $v^(t) = v(t, g(t))$ for every $t \in T$.*

We recall that to say G is $\mathcal{T}$ -measurable means that for every open subset V of Y , the set $\{t \in T : G(t) \cap V \neq \emptyset\}$ belongs to $\mathcal{T}$. Since G is compact valued, this condition implies the apparently stronger condition that for every closed subset C of Y , the set $\{t \in T : G(t) \cap C \neq \emptyset\}$ belongs to $\mathcal{T}$.

In the dynamic programming literature, theorems of the above type are called Dubins-Savage selection theorems. Theorem A is the culmination of many attempts (see, for instance [4, 11]) to improve on the original result of Dubins and Savage [1]. Many infinite horizon problems of dynamic programming can

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be reduced to a one-day problem by use of the Optimality Equation. Once this is done, the situation becomes like that described in Theorem A. One can think of T as the state space, $G(t)$ as the set of actions available to the player when the system is in the state t , and $v(t, y)$ as the reward when the player chooses action y in state t . Theorem A then states that the player can choose an optimal action in each state in a measurable manner. This optimal rule can then be used every day to define an optimal stationary policy in the infinite horizon problem (see, e.g., [4, 9] for details).

It is not unreasonable to conjecture that Theorem A remains true if condition (1) is replaced by the weaker condition

(1') v is $(\mathcal{T} \times \mathbf{B}_y | \text{Graph}(G))$ -measurable and $v(t, \cdot)$ is upper semicontinuous (u.s.c.) on $G(t)$ for every $t \in T$.

The aim of this article is to give a counterexample to this conjecture, even when T is a separable metric space and $\mathcal{T}$ is its Borel σ -field.

2. CODING OF BOREL SETS

Our basic example will be based on a coding of Borel subsets of $\mathbf{R}$, the real line, due to Solovay [10, pp. 25–28]. The coding will now be described in some detail. The main properties of the coding will be stated in a sequence of lemmas. The proofs of the lemmas can be found in [10].

In what follows, ω will denote the set of natural numbers. Let $\{r_i\}$ be an enumeration of the rationals and let J be the pairing function on $\omega \times \omega$ defined by $J(a, b) = 2^a(2b + 1)$. We define the coding recursively as follows:

1. $\alpha \in \omega^\omega$ codes $[r_i, r_j]$ if $\alpha(0) \equiv 0 \pmod{3}$, $\alpha(1) = i$, and $\alpha(2) = j$.
2. Suppose $\alpha_i \in \omega^\omega$ codes $B_i \subseteq \mathbf{R}$, $i = 0, 1, \dots$; then $\alpha \in \omega^\omega$ codes $U_i B_i$ if $\alpha(0) \equiv 1 \pmod{3}$ and $\alpha(J(a, b)) = \alpha_a(b)$.
3. Suppose $\beta \in \omega^\omega$ codes B , $\alpha(0) \equiv 2 \pmod{3}$, and $\alpha(n + 1) = \beta(n)$. Then α codes B^c , the complement of B .
4. $\alpha \in \omega^\omega$ codes $B \subseteq \mathbf{R}$ only as required by 1–3.

Lemma 1. (a) Every $\alpha \in \omega^\omega$ codes at most one subset of $\mathbf{R}$.

(b) Every Borel subset of $\mathbf{R}$ is coded by some $\alpha \in \omega^\omega$.

(c) If a subset of $\mathbf{R}$ is coded by α , then it is Borel.

Next, we define a function $\Phi: \omega^\omega \times \omega \rightarrow \omega^\omega$, with the property that if α codes a Borel set B , then $\Phi(\alpha, \cdot)$ recovers the Borel sets from which B is constructed. For the definition, we need to fix an enumeration $\{s_n\}$ without repetitions, of finite sequences of natural numbers (including the empty sequence) such that if s_n is an initial segment of s_m , then $n \leq m$. So s_0 is the empty sequence. The definition of $\Phi(\alpha, n)$ will proceed by induction on n .

Set $\Phi(\alpha, 0) = \alpha$, $\alpha \in \omega^\omega$. Let $n > 0$ and suppose that $\Phi(\alpha, m)$ has been defined for all $\alpha \in \omega^\omega$ and all $m < n$. Since $n > 0$, s_n is a finite sequence of positive length k , say. Let s_m be the initial segment of s_n of length $(k - 1)$, so $m < n$; let u be the last coordinate of s_n . Now define for $i \in \omega$

$$\Phi(\alpha, n)(i) = \begin{cases} 0 & \text{if } \Phi(\alpha, m)(0) \equiv 0 \pmod{3}, \\ \Phi(\alpha, m)(J(u, i)) & \text{if } \Phi(\alpha, m)(0) \equiv 1 \pmod{3}, \\ \Phi(\alpha, m)(i + 1) & \text{if } \Phi(\alpha, m)(0) \equiv 2 \pmod{3}. \end{cases}$$

Lemma 2. (a) Φ is Borel measurable.

(b) If α codes a Borel set, then for all n , $\Phi(\alpha, n)$ codes a Borel set.

For $\beta \in \omega^\omega$, define $\bar{\beta} \in \omega^\omega$ so that for every $n \in \omega$,

$$s_{\bar{\beta}(n)} = (\beta(0), \beta(1), \dots, \beta(n-1)).$$

Plainly, the map $\beta \rightarrow \bar{\beta}$ is continuous.

Let $C = \{\alpha \in \omega^\omega : (\forall \beta)(\exists n)\Phi(\alpha, \bar{\beta}(n)) \equiv 0\}$.

Lemma 3. (a) C is a coanalytic subset of ω^ω .

(b) C is the set of all $\alpha \in \omega^\omega$ such that α codes a Borel subset of $\mathbf{R}$.

Define a function $\varphi: \omega \times \omega \rightarrow \omega$ such that $s_{\varphi(n, i)}$ is the sequence s_n followed by i .

Let $E \subseteq \omega^\omega \times \mathbf{R} \times 2^\omega$ be defined as follows:

$$\begin{aligned} (\alpha, x, \gamma) \in E \leftrightarrow & (\forall n)[\{\Phi(\alpha, n)(0) \equiv 0 \pmod{3} \\ & \rightarrow \{\gamma(n) = 1 \leftrightarrow (\exists i)(\exists j)(\Phi(\alpha, n)(1) = i \& \Phi(\alpha, n)(2) = j \& x \in [r_i, r_j])\} \\ & \& (\forall n)[\{\Phi(\alpha, n)(0) \equiv 1 \pmod{3} \rightarrow \{\gamma(n) = 1 \leftrightarrow (\exists i)(\gamma(\varphi(n, i)) = 1)\} \\ & (\forall n)[\{\Phi(\alpha, n)(0) \equiv 2 \pmod{3} \rightarrow \{\gamma(n) = 1 \leftrightarrow \gamma(\varphi(n, 0)) = 0\} \end{aligned}$$

Lemma 4. (a) E is a Borel subset of $\omega^\omega \times \mathbf{R} \times 2^\omega$.

(b) For every $\alpha \in C$ and $x \in \mathbf{R}$, there is a unique $\gamma \in 2^\omega$ such that $(\alpha, x, \gamma) \in E$.

(c) If $\alpha \in C$ and $(\alpha, x, \gamma) \in E$, then $(\forall n)(\gamma(n) = 1 \leftrightarrow x \in \text{the Borel set coded by } \Phi(\alpha, n))$.

This completes our description of Solovay's coding of Borel subsets of $\mathbf{R}$.

3. COUNTEREXAMPLES

We use the notation introduced in the previous section.

Let f be the function on $C \times \mathbf{R}$ into 2^ω guaranteed by Lemma 4(b); that is, for $\alpha \in C$ and $x \in \mathbf{R}$, $f(\alpha, x)$ is the unique γ such that $(\alpha, x, \gamma) \in E$. It follows that

$$\text{Graph}(f) = E \cap (C \times \mathbf{R} \times 2^\omega)$$

so that $\text{Graph}(f)$ is (relatively) Borel in $C \times \mathbf{R} \times 2^\omega$. We will now show that f is not Borel measurable. Towards a contradiction, assume that f is Borel measurable on $C \times \mathbf{R}$. Consider the set

$$F = \{(\alpha, x) \in C \times \mathbf{R} : f(\alpha, x)(0) = 0\}.$$

According to Lemmas 3(b) and 4(c) and the definition of Φ , the condition " $\alpha \in C \& f(\alpha, x)(0) = 0$ " states that x does not belong to the Borel set coded by α . Since f is Borel measurable, F is (relatively) Borel in $C \times \mathbf{R}$ so there must exist a Borel subset D of $\omega^\omega \times \mathbf{R}$ such that $F = D \cap (C \times \mathbf{R})$. Fix a homeomorphism h from the set I of irrationals in $\mathbf{R}$ onto ω^ω . Let $B = \{x \in I : (h(x), x) \in D\}$. Plainly, B is a Borel subset of $\mathbf{R}$. So, by Lemmas 1(b) and 3(b), there is $\alpha^* \in C$ such that α^* codes B . Set $x^* = h^{-1}(\alpha^*)$. Then

$$\begin{aligned} x^* \in B \leftrightarrow (\alpha^*, x^*) \in D \leftrightarrow (\alpha^*, x^*) \in F \leftrightarrow f(\alpha^*, x^*)(0) = 0 \\ \leftrightarrow x^* \notin \text{the Borel set coded by } \alpha^* \leftrightarrow x^* \notin B, \end{aligned}$$

a contradiction!

Since the domain of the function f is coanalytic (Lemma 3(a)) and 2^ω is homeomorphic to the Cantor set, the above construction shows that the Measurable Graph Theorem fails, in general, for a function with coanalytic domain X and values in $[0, 1]$. That is, such a function can have a graph that is a Borel subset of $X \times [0, 1]$, but it is not necessarily the case that the function is Borel measurable. A fortiori, a multifunction with coanalytic domain and nonempty, compact subsets of $[0, 1]$ as values can have a Borel measurable graph without being Borel measurable. This answers a question of some interest to workers in stochastic optimization (see, e.g., [6]). On the other hand, it is a classical result (see [3]) that Borel measurability of the graph of a function with analytic domain and values in a Polish space implies the Borel measurability of the function. This is also true, courtesy of the Kunugui-Novikov theorem [3], of a multifunction with analytic domain and nonempty, compact values in a Polish space.

It is now easy to turn the basic example into the desired counterexample to Theorem A with condition (1) replaced by (1').

Let $T = C \times \mathbf{R}$, $\mathcal{F}$ = the Borel σ -field of $C \times \mathbf{R}$, $Y = 2^\omega$, $G(t) = 2^\omega$ for every $t \in T$, and

$$v((\alpha, x), \gamma) = \begin{cases} 1 & \text{if } f(\alpha, x) = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Then v is Borel measurable on $\text{Graph}(G) = T \times Y$, since $\text{Graph}(f)$ is a Borel subset of $T \times Y$. Moreover for fixed $t \in T$, $v(t, \cdot)$ is u.s.c. on $G(t)$. Thus, condition (1') is satisfied. But, plainly, there is no Borel measurable selector g of G such that $v^*(t) = v(t, g(t))$ for all $t \in T$.

In the example above, the function v^* , being identically equal to one, is $\mathcal{F}$ -measurable. We now modify the example so that v^* is not $\mathcal{F}$ -measurable.

Fix a Borel isomorphism of the Borel set E and the interval $[1/2, 1]$ and denote by ψ the restriction of the Borel isomorphism to $\text{Graph}(f)$, so that ψ continues to be a Borel isomorphism of $\text{Graph}(f)$ and the range of ψ . As before, take $T = C \times \mathbf{R}$, $\mathcal{F}$ = the Borel σ -field of $C \times \mathbf{R}$, $Y = 2^\omega$, $G(t) = 2^\omega$ for every $t \in T$, and

$$v((\alpha, x), \gamma) = \begin{cases} \psi(\alpha, x, \gamma) & \text{if } f(\alpha, x) = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily checked that condition (1') is satisfied. Next observe that

$$v^*((\alpha, x)) = \psi(\alpha, x, f(\alpha, x))$$

for all $(\alpha, x) \in T$. If v^* were $\mathcal{F}$ -measurable, then the map $(\alpha, x) \rightarrow (\alpha, x, f(\alpha, x))$ would be $\mathcal{F}$ -measurable, since ψ is a Borel isomorphism. But then f would be $\mathcal{F}$ -measurable, a contradiction!

We remark that if T is an analytic set and $\mathcal{F}$ the Borel σ -field of T , then condition (1') implies condition (1) and so Theorem A is true under condition (1'). To see that (1') implies (1), fix a metric ρ on Y and define for $n = 1, 2, \dots$,

$$v_n(t, y) = \sup_{y' \in G(t)} \{v(t, y') - n\rho(y', y)\}, \quad (t, y) \in \text{Graph}(G).$$

The only nontrivial thing to check regarding condition (1) is that v_n is $(\mathcal{F} \times B_Y | \text{Graph}(G))$ -measurable and this follows from the Kunugui-Novikov theorem

[3]. The validity of Theorem A under condition (1') when T is analytic and $\mathcal{F}$ is its Borel σ -field was observed by Himmelberg et al. [2] and also by Schäl [11].

4. PROOF OF THEOREM A

For the sake of completeness, we give a quick proof of Theorem A. Our proof differs from those of Schäl [8, 9] and Reider [7]. We start with an easy lemma.

Lemma B. *Let Z be a compact metric space. Suppose $f, f_m: Z \rightarrow \mathbf{R}$ are u.s.c. and $f_m \downarrow f$. If $z_m \rightarrow z_0$ in Z , then $\limsup_m f_m(z_m) \leq f(z_0)$.*

Proof. Let $\varepsilon > 0$. Choose a continuous function $g: Z \rightarrow \mathbf{R}$ such that $g \geq f$ and $f(z_0) \geq g(z_0) - \varepsilon$.

Set $g_m = \max(f_m, g)$, $m \geq 1$. Then g_m is u.s.c. and $g_m \downarrow g$. Dini's theorem now applies to yield that the convergence $g_m \downarrow g$ is uniform on Z . Hence

$$\limsup_m f_m(g_m) \leq \lim g_m(z_m) = g(z_0) \leq f(z_0) + \varepsilon.$$

Proof of Theorem A. By [5, Theorem 5], for each $n \geq 1$, there is a $\mathcal{F}$ -measurable selector $g_n: T \rightarrow Y$ of G such that

$$v_n(t, g_n(t)) = \sup_{y \in G(t)} v_n(t, y)$$

for every $t \in T$.

For $t \in T$ set $H(t) = \{y \in G(t) : \text{there is a subsequence } \{g_{n_i}(t)\} \text{ such that } g_{n_i}(t) \rightarrow y\}$. Plainly, $H(t)$ is nonempty, compact as $G(t)$ is. We claim that H is $\mathcal{F}$ -measurable. Let C be closed in Y . Then

$$\{t \in T : H(t) \cap C \neq \emptyset\} = \bigcap_{k \geq 1} \bigcup_{m > k} \left\{ t \in T : \rho(g_m(t), C) < \frac{1}{k} \right\},$$

where ρ is a metric on Y . Since the set on the right belongs to $\mathcal{F}$, H is $\mathcal{F}$ -measurable. Use the Kuratowski-Ryll-Nardzewski selection theorem [3] to get a $\mathcal{F}$ -measurable selector $g: T \rightarrow Y$ of H .

To complete the proof, fix $t \in T$ and let $y^* = g(t)$. So there is a subsequence $\{g_{n_i}(t)\}$ such that $y_{n_i} = g_{n_i}(t) \rightarrow y^*$. By condition (1) and Lemma B, we have

$$\lim_i v_{n_i}(t, y_{n_i}) \leq v(t, y^*).$$

It follows that $v^*(t) = v(t, g(t))$.

REFERENCES

1. L. E. Dubins and L. J. Savage, *Inequalities for stochastic Processes*, Dover, New York, 1976.
2. C. J. Himmelberg, T. Parthasarathy, and F. S. Van Vleck, *Optimal plans for dynamic programming*, Math. Oper. Res. **1** (1976), 390–394.
3. K. Kuratowski and A. Mostowski, *Set theory*, 2nd rev. ed., North-Holland, Amsterdam, 1976.
4. A. Maitra, *Discounted dynamic programming on compact metric spaces*, Sankhyā A **30** (1968), 211–216.
5. A. Maitra and B. V. Rao, *Generalizations of Castaing's theorem on selectors*, Colloq. Math. **44** (1981), 99–104.

6. A. S. Nowak and T. E. S. Raghavan, *Existence of stationary correlated equilibria with symmetric information for discounted stochastic games*, preprint, 1989.
7. U. Rieder, *Measurable selection theorems for optimization problems*, Manuscripta Math. **24** (1978), 115–131.
8. M. Schäl, *A selection theorem for optimization problems*, Arch. Math. **25** (1974), 219–224.
9. ———, *Conditions for optimality in dynamic programming and for the limit of n -state optimal policies to be optimal*, Z. Wahrsch. Verw. Gebiete **32** (1975), 179–196.
10. R. M. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. of Math. (2) **92** (1970), 1–56.
11. D. J. Wagner, *Survey of measurable selection theorems*, SIAM J. Control Optim. **15** (1977), 859–903.

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