NOT ALL σ -COMPLETE BOOLEAN ALGEBRAS ARE QUOTIENTS OF COMPLETE BOOLEAN ALGEBRAS

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ABSTRACT. In Shelah's model of no Borel lifting of the measure algebra we show that there is a σ -complete Boolean algebra of cardinality 2^ω that is not a quotient of a complete Boolean algebra. By Stone duality, there is a basically disconnected space of weight 2^ω that cannot be embedded into an extremally disconnected space.

1. INTRODUCTION

Every compact subspace of an extremally disconnected space is a compact F -space. Dually, every homomorphic image of a complete Boolean algebra is weakly countably complete (abbreviated WCC). The class of extremally disconnected spaces is a very important class of spaces, hence the problem of determining the class of closed subspaces is very fundamental. Obviously it is equally fundamental to investigate the class of quotients of complete Boolean algebras. We will deal with Boolean algebras since the Boolean algebraic notions of completeness and σ -completeness are undoubtedly better known than the topological notions of extremally and basically disconnected spaces.

It was asked by Choquet (see [6]) if every WCC Boolean algebra is a quotient of a complete algebra. It was shown by Louveau [6] that the answer is yes for the case $|B| \leq 2^\omega$ when the continuum hypothesis, CH, is assumed. Van Douwen and van Mill [2] showed that if Martin's Axiom (MA) and $2^\omega = \omega_2$ is assumed, then there is a WCC Boolean algebra of cardinality 2^ω that is not a homomorphic image of even a σ -complete algebra. They asked if every σ -complete algebra is a quotient of a complete algebra. They also remark that this question seems to have first been asked by Koppelberg [5]. The question of the consistency of there being a σ -complete algebra of cardinality 2^ω that is not a quotient of $\mathcal{P}(\omega)$ (the power set of the integers—a complete algebra) is raised again in [7, 4]. Large classes of σ -complete algebras have been shown to be quotients of complete algebras [$\mathfrak{c}$ -field of sets in [10] and the clopen algebras

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of P -spaces (i.e., spaces with the property that all G_δ -sets are open)]. The consistency of all σ -complete Boolean algebras being a quotient of a complete Boolean algebra remains an (interesting) open problem. An intimate connection with Borel liftings is established in [3]. Having made this connection (and having Shelah's model available), we are able to give a very elementary proof of the result.

2. MAIN RESULTS

Recall that the Borel subsets $\mathcal{B}_0$, of the open unit interval is the σ -algebra generated (in the power set) by the open subintervals. A Borel lifting of the measure algebra, is a homomorphism h , from $\mathcal{B}_0$ into $\mathcal{B}_0$, with the property that the kernel $\ker(h)$ is the ideal $\mathcal{I}_{mz}$ of measure zero sets and that b and $h(b)$ differ only by a set of measure zero. The ideal of *meager* or *first category* Borel sets, $\mathcal{I}_{fc}$, is the countably complete ideal generated by the closed sets with empty interior. A Borel lifting of the category algebra (i.e., $\mathcal{B}_0/\mathcal{I}_{fc}$) is defined analogously. We establish the connection between the main problem and liftings.

Theorem 2.1. *If $\mathcal{B}_0$ is a quotient of a complete Boolean algebra then there is a Borel lifting h of the category algebra.*

Proof. Let $\mathcal{B}$ be a complete Boolean algebra and suppose that $\varphi: \mathcal{B} \mapsto \mathcal{B}_0$ is a surjective homomorphism. The proof will be to show that the following diagram can be completed, first with ψ and then r . The lifting h is defined by

$$h(b) = \varphi(r(b/\mathcal{I}_{fc}));$$

in the picture we may think of $h/\mathcal{I}_{fc}$ as $\varphi \circ r$.

$$\begin{array}{ccc} \mathcal{B} & \xleftarrow{\psi} & \text{Free}(\omega) \\ \varphi \downarrow & \swarrow \exists r? & \bigcap \\ \mathcal{B}_0 & \xrightleftharpoons[h]{} & \mathcal{B}_0/\mathcal{I}_{fc} \equiv RO((0, 1)) \end{array}$$

The existence of the free subalgebra in $\mathcal{B}_0/\mathcal{I}_{fc}$ can be deduced from the well-known facts that this is just the regular open algebra (see [1]) and that this is isomorphic to the regular open algebra of the Cantor set 2^ω , which in turn is just the completion of the free algebra with countably many generators. We obtain the map ψ by choosing, for each generator $a \in \text{Free}(\omega)$, a member $b \in \mathcal{B}$ so that $\varphi(b)/\mathcal{I}_{fc} = a$ and by extending by the freeness. Finally, the map r exists because $\mathcal{B}$ is complete, hence injective (see [9, Theorem 33.2]).

Let us now recall the following result of Shelah.

Proposition 2.2 (Shelah [8]). *There is a model in which there is no Borel lifting of the category algebra.*

Therefore the following result has been established.

Corollary 2.3. *There is a model in which there is a σ -complete Boolean algebra of cardinality $\mathfrak{c}$ that is not a quotient of a complete Boolean algebra. Equivalently, there is a model in which there is a compact Basically Disconnected space of weight $\mathfrak{c}$ that cannot be embedded into βN nor any Extremally Disconnected space.*

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