CONVERGENT FREE SEQUENCES IN COMPACT SPACES

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ABSTRACT. We prove that if κ is an uncountable regular cardinal and a compact T_2 space X contains a free sequence of length κ , then X also contains such a sequence that is convergent. This implies that under CH every nonfirst countable compact T_2 space contains a convergent ω_1 -sequence and every compact T_2 space with a small diagonal is metrizable, thus answering old questions raised by the first author and M. Hušek, respectively.

0. INTRODUCTION

While there are compact T_2 spaces with no nontrivial convergent ω -sequences, βN being perhaps the best known such space, every infinite compact T_2 space contains nontrivial convergent transfinite sequences. Indeed, as it is easy to see, if in such a space $\{A_\alpha : \alpha \in \varphi\}$ is a (strictly) decreasing sequence of closed sets with $\bigcap \{A_\alpha : \alpha \in \varphi\} = \{p\}$, a singleton, then for every choice of points $q_\alpha \in A_\alpha \setminus A_{\alpha+1}$, the sequence $\{q_\alpha : \alpha \in \varphi\}$ converges to p , provided that φ is a limit ordinal.

Moreover, it is also easy to see that if the character of a point p in such a space is equal to κ then a decreasing sequence of closed sets $\{A_\alpha : \alpha \in \kappa\}$ does exist with

$$\bigcap \{A_\alpha : \alpha \in \kappa\} = \{p\},$$

hence p is the limit of a κ -sequence. Consequently, since every infinite compact T_2 space has a separable closed infinite subspace, in which then every point has character $\leq c = 2^\omega$, we get that every such space has convergent sequences of length $\leq c$.

On the other hand, it has also been known (see [BSV] or [vM]) that for example, βN contains a convergent sequence of length ω_1 , *without assuming* CH, hence the natural question arises whether every infinite compact T_2 space contains a convergent ω or ω_1 sequence? This question was first formulated by Hušek in the late seventies, and a related stronger problem was independently raised by the first author around the same time: Does every nonfirst countable compact T_2 space contain a convergent ω_1 sequence?

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In this paper we show that every compact T_2 space of uncountable tightness contains a convergent ω_1 sequence, moreover assuming CH every nonfirst countable compact T_2 space does. Also we obtain that under CH every compact T_2 space with a small diagonal is metrizable, thus solving another problem of Hušek from [H]. More generally, using some results of Dow from [D1] we obtain that these consequences will hold in any generic extension obtained by adding Cohen reals to any ground model satisfying CH.

Notation and terminology in this paper will follow that of [J1].

1. THE MAIN THEOREM

The main result of our paper says that if κ is any uncountable regular cardinal and X is a compact T_2 space containing a free sequence of length κ , then X also contains a convergent such sequence. Let us note that having a free sequence $\{p_\alpha : \alpha \in \kappa\}$ converging to p in X also yields a closed set $F \subset X$ with $\chi(p, F) = \kappa$. Indeed, $F = \overline{\{p_\alpha : \alpha \in \kappa\}}$ works since the sets $\overline{\{p_\alpha : \alpha \in \kappa \setminus \beta\}}$ are clopen in F for all $\beta \in \kappa$, for the sequence $\{p_\alpha : \alpha \in \kappa\}$ is free, i.e.,

$$\overline{\{p_\alpha : \alpha \in \beta\}} \cap \overline{\{p_\alpha : \alpha \in \kappa \setminus \beta\}} = \emptyset$$

for $\beta \in \kappa$.

The proof of the main result is split into two cases according to whether X contains a closed set that maps continuously onto 2^κ or not. In the first case we get somewhat stronger results and now we turn to their discussion.

1.1. Theorem. *Let κ be any uncountable cardinal and $f: X \rightarrow 2^\kappa$ be an irreducible continuous map of the compact T_2 space X onto 2^κ . Then for every point $x \in X$ we have*

(i) *there is a (relatively) discrete subspace $D \subset X \setminus \{x\}$ with $|D| = \kappa$ that "converges" to x in the strong sense that every neighbourhood of x contains all but countably many elements of D ;*

(ii) *moreover, if $cf(\kappa) > \omega$, then there is a free sequence $\{x_\alpha : \alpha \in \kappa\} \subset X \setminus \{x\}$ that converges to x .*

Proof. We need several simple facts concerning irreducible maps that are probably well known, although we have not found them in the literature. For the sake of completeness we present them here with their proofs.

First, the image of a regular closed set (in particular, of a clopen set) A under an irreducible closed map $f: X \rightarrow Z$ is regular closed. Indeed, if we had $y \in f[A] \setminus \overline{\text{Int } f[A]} \neq \emptyset$ then $A \setminus f^{-1}[\overline{\text{Int } f[A]}] \neq \emptyset$, hence

$$G = \text{Int } A \setminus f^{-1}[\overline{\text{Int } f[A]}] \neq \emptyset$$

because A is regular closed. Since f is irreducible we have $\text{Int } f[G] \neq \emptyset$, contradicting that both $\text{Int } f[G] \subset \text{Int } f[A]$ and $f[G] \cap \text{Int } f[A] = \emptyset$.

Next we show that if f is as above and A, B are regular closed sets in X , then

$$\text{Int } f[A \cap B] = \text{Int } f[A] \cap \text{Int } f[B].$$

Indeed, assume that $\text{Int } f[A \cap B] \subsetneq \text{Int } f[A] \cap \text{Int } f[B]$. Then $C = \text{Int } f[A] \cap \text{Int } f[B] \setminus \text{Int } f[A \cap B] \neq \emptyset$, hence $D = \text{Int } A \cap f^{-1}[C] \neq \emptyset$, as well. But then $D \cap B = \emptyset$ and $f[D] \subset C \subset f[B]$; consequently, $f[X \setminus D] = Y$, contradicting that f is irreducible.

Let us now return to the proof of the theorem, where, for technical reasons, we first assume that X is also 0-dimensional.

For $S \subset \kappa$ a subset H of 2^κ is called S -determined if $z \in H$ implies $z' \in H$ whenever $z' \in 2^\kappa$ and $z' \upharpoonright S = z \upharpoonright S$. It is well known (cf. [E]) that every regular closed (or open) subset of 2^κ is S -determined for some countable $S \subset \kappa$. Moreover, it is obvious that if H is S -determined then so are $\overline{H}$ and $\text{Int } H$ as well.

For $\alpha \in \kappa$ and $i \in 2$ we shall put

$$U_{\alpha,i} = \{z \in 2^\kappa : z(\alpha) = i\},$$

so the $U_{\alpha,i}$ form the canonical subbase for the product topology on 2^κ .

Now, for any subset $S \subset \kappa$ we let $\mathcal{A}_S$ be the collection of all those clopen subsets A of X that contain x and whose f -image $f[A]$ is S -determined. According to our preliminary remarks, for every clopen neighbourhood A of x there is some countable set $S \subset \kappa$ such that $A \in \mathcal{A}_S$. Moreover, each collection $\mathcal{A}_S$ is closed under finite intersections. Finally, we set $\bigcap \mathcal{A}_S = F_S$. Then F_S is a closed set containing x , and clearly $S \subset T$ implies $F_S \supset F_T$.

Let us put $f(x) = y$ and for any $S \subset \kappa$ we set

$$\Phi_S = \{z \in 2^\kappa : z \upharpoonright S = y \upharpoonright S\}.$$

We claim that $f[F_S] = \Phi_S$ holds for every $S \subset \kappa$. Indeed, first note that

$$f[F_S] = f\left[\bigcap \mathcal{A}_S\right] = \bigcap \{f[A] : A \in \mathcal{A}_S\}$$

because X is compact and $\mathcal{A}_S$ is closed under finite intersections. But for all $A \in \mathcal{A}_S$ we have $y \in f[A]$ and $f[A]$ is S -determined, hence $\Phi_S \subset f[A]$, i.e., $\Phi_S \subset f[F_S]$. On the other hand, for each $\alpha \in S$ we clearly have $f^{-1}[U_{\alpha,y(\alpha)}] \in \mathcal{A}_S$. Consequently,

$$f[F_S] \subset \bigcap \{U_{\alpha,y(\alpha)} : \alpha \in S\} = \Phi_S,$$

hence $f[F_S] = \Phi_S$.

Now for every $\alpha \in \kappa$ we let y^α be the point of 2^κ that agrees with y for $\beta \in \kappa \setminus \{\alpha\}$ but disagrees with it at α , i.e.,

$$y^\alpha(\beta) = \begin{cases} y(\beta) & \text{if } \beta \neq \alpha, \\ 1 - y(\alpha) & \text{if } \beta = \alpha. \end{cases}$$

Since $\Phi_{\kappa \setminus \{\alpha\}} = \{y, y^\alpha\}$, we may pick the points $x^\alpha \in F_{\kappa \setminus \{\alpha\}}$ such that $f(x^\alpha) = y^\alpha$ for all $\alpha \in \kappa$ and claim that the set $D = \{x^\alpha : \alpha \in \kappa\}$ satisfies (i).

That D is discrete follows immediately from the fact that the set $\{y^\alpha : \alpha \in \kappa\}$ is discrete. Next, if A is any clopen neighbourhood of x then there is a countable set $S \subset \kappa$ with $A \in \mathcal{A}_S$. But then for every $\alpha \in \kappa \setminus S$, we have

$$x^\alpha \in F_{\kappa \setminus \{\alpha\}} \subset F_S \subset A,$$

hence (i) holds because X is 0-dimensional.

As for (ii), let us first define the points $y_\alpha \in 2^\kappa$ for $\alpha \in \kappa$ by the stipulation

$$y_\alpha(\beta) = \begin{cases} y(\beta) & \text{if } \beta \leq \alpha, \\ 1 - y(\beta) & \text{for } \alpha < \beta < \kappa. \end{cases}$$

Clearly $\{y_\alpha : \alpha \in \kappa\}$ is a free sequence in 2^κ . Moreover, $y_\alpha \in \Phi_{\alpha+1}$ for each $\alpha \in \kappa$. Since $f(F_{\alpha+1}) = \Phi_{\alpha+1}$, we can choose points $x_\alpha \in F_{\alpha+1}$ for $\alpha \in \kappa$ such that $f(x_\alpha) = y_\alpha$.

Now $\{x_\alpha : \alpha \in \kappa\}$ is a free sequence in X : if $z \in \overline{\{x_\alpha : \alpha \in \beta\}}$ then $f(z) \in \overline{\{y_\alpha : \alpha \in \beta\}}$ by the continuity of f ; hence $f(z) \notin \{y_\alpha : \alpha \in \kappa \setminus \beta\}$ because $\{y_\alpha\}$ is free; thus $z \notin \overline{\{x_\alpha : \alpha \in \kappa \setminus \beta\}}$.

Finally to see that $\{x_\alpha : \alpha \in \kappa\}$ converges to x it clearly suffices to show that $\bigcap \{F_{\alpha+1} : \alpha \in \kappa\} = \{x\}$ since the $F_{\alpha+1}$ are decreasing and $x_\alpha \in F_{\alpha+1} \setminus F_{\alpha+2}$. But for every clopen neighbourhood A of x there is a countable $S \subset \kappa$ with $A \in \mathcal{A}_S$, hence by $cf(\kappa) > \omega$ there is an $\alpha \in \kappa$ such that $S \subset \alpha$. Consequently, $F_{\alpha+1} \subset F_\alpha \subset F_S \subset A$.

Thus, using again that X is 0-dimensional, (ii) has been established.

Now let us consider the general case with an arbitrary compact T_2 space X . By Alexandrov's well-known theorem (see, e.g., [E, 3.2.2]), there is a 0-dimensional compact T_2 space Z admitting an irreducible map $g: Z \rightarrow X$ onto X . Then the composition $h = f \circ g: Z \rightarrow 2^\kappa$ is also irreducible, hence the above considerations can be applied to h and a fixed point $z \in Z$ with $g(z) = x$. This gives us points $\{z^\alpha : \alpha \in \kappa\}$ with $h(z^\alpha) = y^\alpha$ and $\{z_\alpha : \alpha \in \kappa\}$ with $h(z_\alpha) = y_\alpha$. Now it is straightforward to check that the set $D = \{x^\alpha = g(z^\alpha) : \alpha \in \kappa\}$ and the free sequence $\{x_\alpha = g(z_\alpha) : \alpha \in \kappa\}$ are as required by (i) and (ii), respectively.

Now we turn to the formulation of our main result.

1.2. Theorem. *Let κ be an uncountable regular cardinal. If a compact T_2 space X contains a free sequence $S = \{x_\alpha : \alpha \in \kappa\}$ of length κ then it also contains one that is convergent.*

Proof. Let us first note that the closure $\overline{S}$ of S can be mapped continuously onto the space $\kappa + 1$ taken with its usual order topology. Indeed, if for any limit ordinal $\lambda \leq \kappa$ we set

$$A_\lambda = \bigcap_{\alpha \in \lambda} \overline{\{x_\beta : \beta \in \lambda \setminus \alpha\}},$$

or equivalently, since S is free,

$$A_\lambda = \{x \in X : \lambda = \min\{\alpha : x \in \{x_\beta : \beta \in \alpha\}'\}\},$$

then the map $f: \overline{S} \rightarrow \kappa + 1$ (defined below) is clearly continuous:

$$\begin{aligned} f(x_n) &= n && \text{for } n \in \omega, \\ f(x) &= \lambda && \text{for } \lambda \leq \kappa \text{ limit and } x \in A_\lambda, \\ f(x_\alpha) &= \alpha + 1 && \text{for } \alpha \in \kappa \setminus \omega. \end{aligned}$$

Thus we may assume without any loss of generality that X admits an irreducible map f onto $\kappa + 1$. Moreover, using 1.1(ii), we may also assume that no closed subspace of X can be mapped onto 2^κ . This, however, by Šapirovskii's celebrated result (cf. [J1, 3.18]) implies that every nonempty closed subset F of X has a point p with $\pi\chi(p, F) < \kappa$.

For every ordinal $\alpha \leq \kappa$ let us define $Z_\alpha = f^{-1}[\kappa + 1 \setminus \alpha]$. In particular, then the set $Z_\kappa = f^{-1}[\{\kappa\}]$ is nowhere dense in X because f is irreducible.

Let $p \in Z_\kappa$ be any point having π -character $\pi\chi(p, Z_\kappa) = \pi$ less than κ in Z_κ . We are going to show that there is a free sequence of length κ converging to p . This will clearly establish our result.

First we claim that for every open subset G of X if $G \cap Z_\kappa \neq \emptyset$ then $f[G] \cap \kappa$ is cofinal in κ . Indeed, otherwise we had an $\alpha \in \kappa$ such that $G \setminus Z_\kappa \subset f^{-1}[\alpha]$, hence as Z_κ is nowhere dense we had

$$G \subset \overline{G \setminus Z_\kappa} \subset \overline{f^{-1}[\alpha]} \subset f^{-1}[\alpha + 1],$$

contradicting that $G \cap Z_\kappa \neq \emptyset$.

Using $\pi\chi(p, Z_\kappa) = \pi$ ($< \kappa$), we may choose open sets $\{G_\gamma : \gamma \in \pi\}$ in X such that $\{G_\gamma \cap Z_\kappa : \gamma \in \pi\}$ forms a local π -base at p in Z_κ . Let us pick a point $q_\gamma \in G_\gamma \cap Z_\kappa$ for each $\gamma \in \pi$ and then choose the open set B_γ in X such that $q_\gamma \in B_\gamma \subset \overline{B_\gamma} \subset G_\gamma$. By the above claim, then

$$C_\gamma = \kappa \cap f[\overline{B_\gamma}]$$

is a closed unbounded subset of κ , hence by $\pi < \kappa$ so is $C = \bigcap \{C_\gamma : \gamma \in \pi\}$. (Here we use the regularity of κ .)

Now, for every $\alpha \in \kappa$ we let $\mathcal{V}_\alpha$ be the collection of all those open neighbourhoods V of p for which $\overline{B_\gamma} \cap Z_\kappa \subset V$ implies $\overline{B_\gamma} \cap Z_\alpha \subset V$ for all $\gamma \in \pi$. Clearly $V_1, V_2 \in \mathcal{V}_\alpha$ implies $V_1 \cap V_2 \in \mathcal{V}_\alpha$. Moreover $\mathcal{V}_\alpha \subset \mathcal{V}_\beta$ if $\alpha < \beta$. Let us put

$$F_\alpha = \bigcap \{\overline{V} : V \in \mathcal{V}_\alpha\}.$$

Then we have $F_\alpha \supset F_\beta$ for $\alpha < \beta$ and

$$f[F_\alpha] = \bigcap \{f[\overline{V}] : V \in \mathcal{V}_\alpha\}$$

by our above remarks.

But for every $V \in \mathcal{V}_\alpha$ there is a $\gamma \in \pi$ such that $\overline{B_\gamma} \cap Z_\alpha \subset G_\gamma \cap Z_\alpha \subset V$. Consequently,

$$C \setminus \alpha \subset C_\gamma \setminus \alpha \subset f[\overline{B_\gamma} \cap Z_\alpha] \subset f[\overline{V}].$$

Thus we obtain $C \setminus \alpha \subset f[F_\alpha]$ and, in particular, $F_\alpha \neq \{p\}$.

Next we show that for every neighbourhood V of p there is some $\alpha \in \kappa$ with $V \in \mathcal{V}_\alpha$. Indeed, if $\overline{B_\gamma} \cap Z_\kappa \subset V$ then

$$Z_\kappa = \bigcap \{Z_\alpha : \alpha \in \kappa\},$$

and $Z_\alpha \subset Z_\beta$ for $\beta < \alpha$ imply that, as X is compact, there is some $\alpha_\gamma \in \kappa$ such that $\overline{B_\gamma} \cap Z_{\alpha_\gamma} \subset V$. Thus if $\alpha \in \kappa$ is chosen with $\alpha_\gamma \leq \alpha$ for each $\gamma \in \pi$, and this is possible since κ is regular and $\pi < \kappa$, then $V \in \mathcal{V}_\alpha$ indeed.

Putting all this together we obtain from the sets F_α a strictly decreasing sequence of closed sets whose intersection is $\{p\}$. By suitably thinning out, we may assume that $F_{\alpha+1} \subsetneq F_\alpha$ for all $\alpha \in \kappa$; hence if $p_\alpha \in F_{\alpha+1} \setminus F_\alpha$ then the sequence $\{p_\alpha : \alpha \in \kappa\}$ converges to p .

Finally, by passing to a subsequence we may assume that if $\alpha < \beta$ then $f(p_\alpha) < f(p_\beta)$, and then the subsequence $\{p_{\alpha+1} : \alpha \in \kappa\}$ is also free because its f -image $\{f(p_{\alpha+1}) : \alpha \in \kappa\}$ is.

2. APPLICATIONS

In this section we are going to present several results that are more or less immediate consequences of our main theorem and other known results. To start with, we consider a result which, for $\kappa = \omega$, was formulated in the introduction.

2.1. Corollary. *If $2^\kappa = \kappa^+$ and X is a compact T_2 space of character $\chi(X) > \kappa$, then X has a closed subset F with a point p such that $\chi(p, F) = \kappa^+$, hence a convergent κ^+ -sequence as well.*

Proof. If $t(X) > \kappa$ then this is immediate from 1.2, even without assuming $2^\kappa = \kappa^+$. (Remember that a convergent free κ^+ -sequence yields F and p as required.)

Now if $t(X) \leq \kappa$ then for any point $p \in X$ with $\chi(p, X) \geq \kappa^+$ we may apply 6.14(b) of [J1] with $\lambda = \kappa^+$ to obtain a set $Y \subset X$ with $|Y| \leq \kappa^+$ such that

$$\chi(p, Y) = \chi(p, \overline{Y}) \geq \kappa^+.$$

If $|Y| \leq \kappa$ then we have $w(\overline{Y}) \leq 2^{d(\overline{Y})} \leq 2^\kappa = \kappa^+$, hence we must have $\chi(p, \overline{Y}) = \kappa^+$.

If, on the other hand, $|Y| = \kappa^+$ then let us write $Y = \bigcup \{Y_\alpha : \alpha \in \kappa^+\}$ with $|Y_\alpha| = \kappa$ and $Y_\alpha \subset Y_\beta$ for $\alpha, \beta \in \kappa^+$ and $\alpha < \beta$. Since $t(X) \leq \kappa$, we also have $\overline{Y} = \bigcup \{\overline{Y}_\alpha : \alpha \in \kappa^+\}$. But then for each $\alpha \in \kappa^+$ we have $w(\overline{Y}_\alpha) \leq \kappa^+$ as above, hence (cf. [J1]) we have

$$w(\overline{Y}) = nw(\overline{Y}) \leq \sum \{nw(\overline{Y}_\alpha) : \alpha \in \kappa^+\} \leq \kappa^+,$$

and thus again $\chi(p, \overline{Y}) = \kappa^+$.

For the case $\kappa = \omega$ we can prove the following partial strengthening of 2.1.

2.2. Corollary. *If V satisfies CH and W is an extension of V obtained by adding some Cohen reals to V , then in W every nonfirst countable compact T_2 space X has a convergent ω_1 -sequence. (Note that $W = V$ is permitted here.)*

Proof. As above, using our main theorem we can assume that $t(X) = \omega$, hence, by 5.8(b) of [D1], every point $p \in X$ with $\chi(p, X) > \omega$ is the limit of an ω_1 -sequence.

In [H] Hušek introduced the notion of small diagonal and asked whether CH implies that every compact T_2 space with a small diagonal is metrizable. (Recall that X is said to have small diagonal if for every uncountable set $H \subset X^2 \setminus \Delta$ there is a neighbourhood U of Δ with $H \setminus U$ also uncountable. In other words, this means that no ω_1 -sequence from $X^2 \setminus \Delta$ can converge to Δ .) We can now establish the following stronger result.

2.3. Corollary. *If W is as in 2.2 then in W every compact T_2 space X with a small diagonal is metrizable.*

Proof. By 5.8(c) of [D1] this is so if $t(X) \leq \omega$. If on the other hand $t(X) > \omega$, then by 1.2 there is a convergent ω_1 -sequence $\{p_\alpha : \alpha \in \omega_1\}$ in X consisting of distinct points. But if $\langle p_\alpha \rangle$ converges to p then $\{\langle p_\alpha, p_{\alpha+1} \rangle : \alpha \in \omega_1\} \subset X^2 \setminus \Delta$ converges to $\langle p, p \rangle \in \Delta$, which shows that X cannot have small diagonal.

Let us now recall from the introduction Hušek's other conjecture saying that every compact T_2 space contains a convergent ω - or ω_1 -sequence. Our next result shows that this is implied by a conjecture formulated by the first author in [J2], namely, that every compact T_2 space of countable tightness has a point of character $\leq \omega_1$. In fact, a little more is true.

2.4. Corollary. *If every compact T_2 space of countable tightness has a point of character $\leq \omega_1$ then in every infinite compact T_2 space X there is a closed set F with a point $p \in F$ such that $\chi(p, F) = \omega$ or $\chi(p, F) = \omega_1$.*

Proof. If $t(X) > \omega$ then we can simply apply 1.2. Next if $t(X) = \omega$ and X is also scattered, then X contains the one-point compactification of an infinite set, hence a closed set F with a point p satisfying $\chi(p, F) = \omega$. Finally if $t(X) = \omega$ and X is not scattered, then X contains a closed set F that is dense in itself and, by our assumption, for some point $p \in F$ we have $\chi(p, F) = \omega$ or $\chi(p, X) = \omega_1$.

In [J3] it was shown that the assumption of 2.4 is valid in a generic extension obtained by adding ω_1 Cohen reals to an arbitrary ground model. Moreover, Dow proved in [D2] that this assumption, hence by 2.4 also Hušek's conjecture, is valid under PFA.

Next we give a very simple and quick proof of a result from [J4]. In order to formulate this we recall that a space X is said to omit a cardinal λ if $|X| > \lambda$ but X does not contain a closed subset of size λ .

2.5. Corollary. *If $2^{2^\kappa} = \kappa^{++}$ then no compact space X may omit both κ^+ and κ^{++} .*

Proof. Let $|X| > \kappa^{++}$. If $t(X) \leq \kappa$ then for every $Y \subset X$ with $|Y| = \kappa^{++}$ we have $|\overline{Y}| = \kappa^{++}$ because

$$\overline{Y} = \bigcup \{ \overline{Z} : Z \subset Y \text{ and } |Z| \leq \kappa \}.$$

If, however, $t(X) > \kappa$, then by 1.2 there is a convergent κ^+ -sequence $S = \{x_\alpha : \alpha \in \kappa^+\}$ in X . Then we have

$$\overline{S} = \bigcup \{ \overline{\{x_\beta : \beta \in \alpha\}} : \alpha \in \kappa^+ \} \cup \{x\},$$

where x is the limit of S , hence clearly either $|\overline{S}| = \kappa^+$ or $|\overline{S}| = \kappa^{++}$.

Note that the same proof actually yields in ZFC that no compact T_2 space can omit every cardinal in the interval $[\kappa^+, 2^{2^\kappa}]$.

We finish with a result that is not connected with our main theorem, still we thought to add it here because it is directly related to Hušek's conjecture. This result uses the set theoretic principle $\clubsuit$ (club), introduced by Ostaszewski in [O], saying that with every limit ordinal $\lambda \in \omega_1$ we can associate an ω -type subset $S_\lambda \subset \lambda$ with $\bigcup S_\lambda = \lambda$ such that for every uncountable set $H \subset \omega_1$ there is a λ with $S_\lambda \subset H$. $\clubsuit$ is known to be consistent with the continuum being arbitrarily large.

2.6. Theorem. *Assume $\clubsuit$ and let X be a countably compact infinite T_2 space that contains no nontrivial convergent ω -sequence. Then there is an ω_1 -size supspace $Y = \{x_\alpha : \alpha \in \omega_1\}$ of X such that in Y every (relatively) open set is either countable or co-countable.*

Proof. Let $\langle S_\lambda : \lambda \in L_1 \rangle$ be a $\clubsuit$ -sequence with L_1 denoting the set of limit ordinals in ω_1 . By transfinite induction on $\alpha \in \omega_1$, we are going to define infinite closed sets $F_\alpha \subset X$ and points $x_\alpha \in F_\alpha$ as follows.

Let us put $F_0 = X$, and if the infinite closed set F_α has been defined then we pick $x_\alpha \in F_\alpha$ and $F_{\alpha+1} \subset F_\alpha \setminus \{x_\alpha\}$ such that $F_{\alpha+1}$ be closed and infinite.

If $\lambda \in L_1$ and the F_α have been defined for all $\alpha \in \lambda$ then $\langle F_\alpha : \alpha \in S_\lambda \rangle$ is a decreasing ω -sequence of closed sets, hence the set $F_\lambda = \{x_\alpha : \alpha \in S_\lambda\}'$ of all accumulation points of the sequence $\{x_\alpha : \alpha \in S_\lambda\}$ is nonempty by countable compactness. Moreover, it is infinite because X contains no convergent ω -sequence. This completes the induction.

It is immediate from our construction that for every $\lambda \in L_1$ we have

$$\{x_\alpha : \alpha \in \omega_1 \setminus \lambda\} \subset \overline{\{x_\beta : \beta \in S_\lambda\}}.$$

Consequently, since for every uncountable set $Z \subset Y$ there is a $\lambda \in L_1$ with $\{x_\beta : \beta \in S_\lambda\} \subset Z$, any uncountable closed (or open) subset of Y must be co-countable. The proof is thus completed.

Let us first note that Y is an S -space if it is also T_3 . Moreover it is obvious that Y can have at most one complete accumulation point. Consequently, we have

2.7. Corollary. *If $\clubsuit$ holds then every infinite initially ω_1 -compact T_2 -space has a nontrivial convergent ω or ω_1 -sequence.*

REFERENCES

- [BSV] B. Balcar, P. Simon, and P. Vojtáš, *Refinement properties and extensions of filters in Boolean algebras*, Trans. Amer. Math. Soc. **267** (1981), 265–283.
- [D1] A. Dow, *Compact spaces of countable tightness in the Cohen model*, Set Theory and its Appl. (J. Steprāns and S. Watson, eds.), Lecture Notes in Math., vol. 1401, Springer-Verlag, New York, 1989, pp. 55–67.
- [D2] —, *An introduction to applications of elementary submodels in topology*, Topology Proc. **13** (1988), 17–72.
- [E] R. Engelking, *General topology*, Helderman Verlag, Berlin, 1989.
- [H] M. Hušek, *Topological spaces without κ -accessible diagonal*, Comment. Math. Univ. Carolin. **18** (1977), 777–788.
- [J1] I. Juhász, *Cardinal function—ten years later*, Math. Centre Tract, vol. 123, North-Holland, Amsterdam, 1980.
- [J2] —, *A weakening of club, with applications to topology*, Comment. Math. Univ. Carolin. **29** (1988), 767–773.
- [J3] —, *On the minimum character of points in compact space*, Proc. 1989 Topology Conf. Pécs (to appear).
- [J4] —, *Two set-theoretic problems in topology*, Proc. Fourth Prague Sympos. on Gen. Topology Part A, Springer-Verlag, New York, 1977, pp. 115–123.
- [vM] J. van Mill, *An introduction to $\beta\omega$* , Handbook of Set-Theoretic Topology, 1984, pp. 503–567.
- [O] A. Ostaszewski, *On countably compact, perfectly normal spaces*, J. London Math. Soc. (2) **14** (1976), 505–516.

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