A SHORT PROOF OF THE FROBENIUS THEOREM

ALBERT T. LUNDELL

(Communicated by Jonathan M. Rosenberg)

ABSTRACT. By separating the algebraic and analytic aspect of Frobenius' theorem on involutive distributions, we are able to give a simplified proof.

The Frobenius theorem may be stated as follows [B, p. 161] :

Theorem. *An r -distribution Δ on an m -manifold M is involutive if and only if Δ is completely integrable.*

We recall that Δ is involutive if in the neighborhood of each point of M there is a local basis of vector fields $\{X_1, X_2, \dots, X_r\}$ such that

$$[X_i, X_j] = \sum_{k=1}^r c_{i,j}^k X_k \quad \text{for } 1 \leq i, j \leq r,$$

and Δ is completely integrable if for each point $p \in M$ there is a coordinate system $u: (U, p) \rightarrow (\mathbb{R}^m, 0)$ such that Δ is spanned by $\{\partial/\partial u^1, \dots, \partial/\partial u^r\}$.

Clearly, if Δ is completely integrable then Δ is involutive. Thus we assume that Δ is involutive.

The following proof separates the algebraic and analytic aspects of the theorem. The algebraic part is completely elementary; we give a reference for the analytic part.

Lemma 1. *There is a coordinate system $v: (V, p) \rightarrow (\mathbb{R}^m, 0)$ such that Δ has a local basis $\{X_1, \dots, X_r\}$ of the form*

$$X_i = \frac{\partial}{\partial v^i} + \sum_{j=r+1}^m b_i^j \frac{\partial}{\partial v^j} \quad \text{for } 1 \leq i \leq r.$$

Proof. Choose coordinates $v: (V', p) \rightarrow (\mathbb{R}^m, 0)$ and let $\{Y_1, \dots, Y_r\}$ be a local basis of vector fields for Δ in V' . Then

$$Y_i = \sum_{j=1}^m a_i^j \frac{\partial}{\partial v^j} \quad \text{for } 1 \leq i \leq r,$$

where the functions a_i^j are C^∞ on V' . Since the Y_i are independent, the $r \times m$ matrix $A = (a_i^j)$ is of rank r , and without loss of generality, we may assume

Received by the editors April 29, 1991.

1991 *Mathematics Subject Classification.* Primary 53C99.

that in a (possibly smaller) neighborhood V of p the $r \times r$ submatrix $A' = (a_i^j)$, $1 \leq i, j \leq r$, is nonsingular. If $A'^{-1} = (\hat{a}_i^j)$, we set $X_i = \sum_{j=1}^r \hat{a}_i^j Y_j$. $\square$

Corollary. *If the r -distribution Δ is involutive, then the local basis of the lemma satisfies $[X_i, X_j] = 0$ for $1 \leq i, j \leq r$.*

Proof. Observe that

$$[X_i, X_j] \in \text{span} \left\{ \frac{\partial}{\partial v^{r+1}}, \dots, \frac{\partial}{\partial v^m} \right\} \quad \text{for } 1 \leq i, j \leq r.$$

On the other hand, since Δ is involutive, $[X_i, X_j] \in \text{span}\{X_1, \dots, X_r\}$ for $1 \leq i, j \leq r$. Thus

$$[X_i, X_j] \in \text{span}\{X_1, \dots, X_r\} \cap \text{span} \left\{ \frac{\partial}{\partial v^{r+1}}, \dots, \frac{\partial}{\partial v^m} \right\} = \{0\}. \quad \square$$

For a proof of the theorem, we now appeal to [BC, Theorem 5, p. 20], which states that if $\{X_1, \dots, X_r\}$ is a set of independent vector fields on an open set V such that $[X_i, X_j] = 0$ on W , then for $p \in W$ there is a coordinate system $u: (U, p) \rightarrow (\mathbb{R}^m, 0)$ such that $X_i = \partial/\partial u^i$ for $1 \leq i \leq r$. $\square$

For a proof of the theorem using a more elementary analytic result, we quote the familiar [W, Proposition 1.53, p. 40], which follows directly from an existence theorem in ordinary differential equations. We state this as a lemma and do not provide a proof.

Lemma 2. *If X is a vector field with $X(p) \neq 0$, then there is a coordinate system $u: (U, p) \rightarrow (\mathbb{R}^m, 0)$ such that on U , $X = \partial/\partial u^1$.* $\square$

Of course this is the case $r = 1$ of [BC]. The following proof of the Frobenius Theorem is similar to that of [CW].

The proof is by induction on r . The case $r = 1$ is just Lemma 2. Suppose that $r \geq 2$ and the result is true for $r - 1$. Using the corollary above, we may assume that in a neighborhood of p the distribution Δ has a basis $\{X_1, \dots, X_r\}$ satisfying $[X_i, X_j] = 0$. The inductive hypothesis enables us to choose coordinates $v: (V, p) \rightarrow (\mathbb{R}^m, 0)$ so that on V , we have

$$\text{span}\{X_1, \dots, X_{r-1}\} = \text{span} \left\{ \frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^{r-1}} \right\}.$$

Since the $\partial/\partial v^i$ and the X_i are related by a nonsingular linear transformation and $[X_i, X_r] = 0$, we see that

$$\begin{aligned} \left[\frac{\partial}{\partial v^i}, X_r \right] &\in \text{span}\{X_1, \dots, X_{r-1}\} \\ &= \text{span} \left\{ \frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^{r-1}} \right\} \quad \text{for } 1 \leq i \leq r-1. \end{aligned}$$

In a neighborhood of p we may write $X_r = \sum_{i=1}^m a^i \partial/\partial v^i$, and the condition on brackets yields

$$\frac{\partial a^i}{\partial v^j} = 0 \quad \text{for } r \leq i \leq m \text{ and } 1 \leq j \leq r-1.$$

Thus for $r \leq i \leq m$, the a^i are independent of $v^1, \dots, v^{r-1}$. If we set

$$Y = X_r - \sum_{i=1}^{r-1} a^i \frac{\partial}{\partial v^i} = \sum_{i=r}^m a^i \frac{\partial}{\partial v^i},$$

then

$$\text{span}\{X_1, \dots, X_r\} = \text{span}\left\{\frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^{r-1}}, Y\right\}$$

and Y depends only on $v^r, \dots, v^m$. Using Lemma 2, we may choose coordinates $u : (U, p) \rightarrow (\mathbb{R}^m, 0)$ of the form

$$u^i = \begin{cases} v^i & \text{for } 1 \leq i \leq r-1, \\ \tilde{u}^i(v^r, \dots, v^m) & \text{for } r \leq i \leq m \end{cases}$$

so that

$$\begin{aligned} \frac{\partial}{\partial u^i} &= \frac{\partial}{\partial v^i} \quad \text{for } 1 \leq i \leq r-1, \\ \frac{\partial}{\partial u^r} &= \frac{\partial}{\partial v^r} + \sum_{k=r+1}^m a^k \frac{\partial}{\partial v^k} = Y. \end{aligned}$$

We have $\{\partial/\partial u^1, \dots, \partial/\partial u^r\}$ as a basis for Δ on U . $\square$

Remark. Either of these proofs will work for the Frobenius theorem with parameters, the only changes being mention of the parameters.

REFERENCES

- [BC] R. L. Bishop and R. J. Crittenden, *Geometry of manifolds*, Academic Press, New York, 1964.
- [B] W. M. Boothby, *An introduction to differentiable manifolds and Riemannian geometry*, Academic Press, Orlando, FL, 1986.
- [CW] S. S. Chern and J. G. Wolfson, *A simple proof of the Frobenius theorem*, Manifolds and Lie Groups, Progr. Math., vol. 14, Birkhauser, Boston, 1981, pp. 67–69.
- [W] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer-Verlag, New York, 1971.

DEPARTMENT OF MATHEMATICS, BOX 426, UNIVERSITY OF COLORADO, BOULDER, COLORADO 80309

E-mail address: lundell@euclid.colorado.edu