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NORMS ON UNITIZATIONS OF BANACH ALGEBRAS

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ABSTRACT. Equivalence of various norms on the unitization of a nonunital Banach algebra is established, with bounds (1 and $6\exp(1)$) uniform over the class of such algebras. A tighter bound, 3, is obtained in C^*-algebras for elements with Hermitian nonunital parts.

The algebra norm $\|\cdot\|$ on a nonunital Banach algebra A can be extended to an algebra norm on the unitization A^+ in many ways. Proposition 4.3 in [3] states that among these extensions, the l_1-norm

$$\|\lambda e + a\|_1 = |\lambda| + \|a\|$$

is maximal and the operator norm

$$\|\lambda e + a\|_{\mathrm{op}} = \sup\{\|\lambda x + ax\| : \|x\| \le 1\}$$

is minimal, provided that it does extend $\|\cdot\|$, i.e., that $\|\cdot\|$ is a regular (= operator) norm.

In the latter case, A^+ is complete under both $\|\cdot\|_1$ and $\|\cdot\|_{\mathrm{op}}$, so by the "two-norm lemma" [2, II.2.5] these two norms are equivalent; the pure existence nature of the lemma does not yield an explicit bound M in $\|\cdot\|_1 \le M\|\cdot\|_{\mathrm{op}}$ and such a bound seems to depend on the algebra A.

The present theorem establishes uniform equivalence of the two unitization norms over the class of nonunital Banach algebras with regular norms.

Theorem. *For every nonunital Banach algebra A with unitization A^+ and with regular norm, and for every $\lambda \in \mathbb{C}$ and $a \in A$, we have*

$$\|\lambda e + a\|_{\mathrm{op}} \le \|\lambda e + a\|_1 \le (6\exp 1)\|\lambda e + a\|_{\mathrm{op}}.$$

If A is a C^*-algebra, $a \in A$ is hermitian, and λ is complex then

$$\|\lambda e + a\|_1 \le 3\|\lambda e + a\|_{\mathrm{op}}$$

and the constant 3 is best (minimal) possible.

Proof. In a general algebra A with a regular norm, we have an extension of the classical inequality for the numerical radius $v(a)$ [1, Theorem 4.1]:

$$v(a) \le \|a\| \le (\exp 1)v(a).$$

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Without loss of generality, assume that $a \neq 0$. We know that the closure K of the numerical range of a in a nonunital algebra contains 0; our first task is to estimate $v(\lambda + a)$ from below: From the geometry of the complex plane we see that the diameter d of the compact K is realized as the distance $d = |\alpha - \beta|$ with α, $\beta \in K$, and comparison with the special case $\lambda_0 = -(\alpha+\beta)/2$ leads to

$$v(\lambda e + a) = \max\{|\lambda + \xi| : \xi \in K\} \geq \tfrac{1}{2} d.$$

Also, since $0 \in K$, we have $d \geq v(a)$. Altogether,

$$v(\lambda e + a) \geq \tfrac{1}{2} v(a).$$

Now we split estimates into cases $|\lambda| \leq 2\|a\|$ and $|\lambda| > 2\|a\|$. The former case gives

$$\frac{\|\lambda e + a\|_{\text{op}}}{|\lambda| + \|a\|} \geq \frac{v(a)/2}{2\|a\| + (\exp 1) v(a)} \geq \frac{v(a)/2}{(3\exp 1) v(a)} = \frac{1}{6 \exp 1};$$

the latter case $|\lambda| > 2\|a\|$ gives, using the triangle inequality and the fact that the fraction in the middle increases with $|\lambda|$,

$$\frac{\|\lambda e + a\|_{\text{op}}}{|\lambda| + \|a\|} \geq \frac{|\lambda| - \|a\|}{|\lambda| + \|a\|} \geq \frac{1}{3}.$$

We conclude that for all complex λ,

$$\|\lambda + a\|_1 \leq (6 \exp 1) \|\lambda + a\|_{\text{op}}.$$

Now the C^*-algebra case: The closure of the numerical range of a Hermitian a is the smallest real interval $[\alpha, \beta]$ containing the spectrum of a, and for all complex λ we have

$$\begin{aligned} \|\lambda e + a\|_1 &= |\lambda| + \max(|\alpha|, |\beta|), \\ \|\lambda e + a\|_{\text{op}} &= \max(|\lambda + \alpha|, |\lambda + \beta|). \end{aligned}$$

The expression to minimize is

$$q(\lambda) = \frac{\max(|\lambda + \alpha|, |\lambda + \beta|)}{|\lambda| + \max(|\alpha|, |\beta|)}.$$

Without loss of generality, we assume that $\alpha \leq 0 < \beta$ and $\gamma = (\alpha + \beta)/2 \geq 0$ (recall that 0 is in the spectrum of a); otherwise we replace a with $-a$.

From now on, this is a problem about complex numbers. We split it into four cases:

(C1) λ real,
(C2) λ not real, $\Re\lambda > -\gamma$,
(C3) λ is not real, $\Re\lambda < -\gamma$,
(C4) λ not real, $\Re\lambda = -\gamma$.

In (C1) q is continuous, piecewise monotone with breakpoints $-\beta$, 0, $-\gamma$, $-\alpha$, and respective values,

$$\frac{\beta + |\alpha|}{2\beta} \geq \frac{1}{2}, \qquad \frac{\beta}{\beta} = 1, \qquad \frac{\beta + |\alpha|}{3\beta - |\alpha|} \geq \frac{1}{3}, \qquad \frac{\beta + |\alpha|}{\beta + |\alpha|} = 1,$$

and q approaches 1 as $|\lambda| \to \infty$. The best we can say about q, therefore, is $q \geq \frac{1}{3}$, attained when $\alpha = 0$.

Case (C4). Write, for symmetry, $\alpha = \gamma - \rho \le 0$, $\beta = \gamma + \rho > 0$, so that $\rho = (\beta - \alpha)/2$. Also, we substitute $p = -\gamma + \sqrt{\gamma^2 + \nu^2}$ (note $p \ge 0$), so that $\nu^2 = p^2 + 2p\gamma$. To prove that $Q(\nu) = q(-\gamma + i\nu) \ge \frac{1}{3}$, write

$$Q(\nu) = \frac{|\lambda + \alpha|}{|\lambda| + \max(|\alpha|, |\beta|)} = \frac{\sqrt{\rho^2 + \nu^2}}{\sqrt{\gamma^2 + \nu^2} + \gamma + \rho},$$

$$Q^2(\nu) - \frac{1}{9} = \frac{9(\rho^2 + p^2 + 2p\gamma) - (p + \rho + 2\gamma)^2}{9(\rho + p + 2\gamma)^2}$$

$$= \frac{2(\rho - p - \gamma)^2 + 6(\rho + \gamma)(\rho - \gamma + p)}{9(\rho + p + 2\gamma)^2} \ge 0$$

since both $\rho + \gamma > 0$ and $\rho - \gamma + p = |\alpha| + p \ge 0$.

Cases (C2) and (C3). Except on the set $\{\lambda | \Re\lambda = -\gamma \text{ or } \lambda = 0\}$, q has a gradient

$$\nabla q(\lambda) = \frac{(\lambda + \beta)(|\lambda| + \beta)/|\lambda + \beta| - |\lambda + \beta|\lambda/|\lambda|}{(|\lambda| + \beta)^2} \quad \text{for } \Re\lambda > -\gamma,$$

$$= \frac{(\lambda + \alpha)(|\lambda| + \beta)/|\lambda + \alpha| - |\lambda + \alpha|\lambda/|\lambda|}{(|\lambda| + \beta)^2}.$$

Remark. The bound $6 \exp 1$ is not the best; by splitting at $(1 + 1/(2\exp 1))\|a\|$ instead of at $2\|a\|$ in the proof, we could reduce the bound $6 \exp 1$ to $1 + 4\exp 1$, but we suspect that even this can be improved.

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