TRICANONICAL SYSTEM OF A SURFACE OF GENERAL TYPE IN POSITIVE CHARACTERISTIC

TOHRU NAKASHIMA

(Communicated by Louis J. Ratliff, Jr.)

ABSTRACT. Using vector bundle method, we study the tricanonical system on a minimal surface of general type defined over an algebraically closed field of positive characteristic. Under some conditions, it is proved that it has no fixed component.

INTRODUCTION

Pluricanonical systems on minimal surfaces of general type have been studied by many authors after the fundamental work of Bombieri [B]. Recently, Ekedahl extended many of the classical results of Bombieri to the positive characteristic case [E]. In particular, he proved that the *m*th pluricanonical system |mK| is base point free if $m \ge 4$ or $m \ge 3$ and $K^2 \ge 2$. Some of the remaining cases are treated in [SB]. In this short note we shall consider the fixed part of the tricanonical system |3K| under certain conditions. Our purpose is to show the following

Theorem. Let X be a minimal surface of general type defined over an algebraically closed field k. Assume char(k) = p > 2 and $K^2 = \chi(\mathscr{O}_X) = 1$. Then |3K| has no fixed component if it is not composed with a pencil.

PROOF OF THE THEOREM

Proposition 1.1. Let X be a minimal surface of general type defined over an algebraically closed field of characteristic p > 2 such that $K^2 = 1$. Then (-2) curves cannot be contained in the fixed part of |3K|.

Proof. Suppose that a (-2) curve C is a fixed component of |3K|. We have the exact sequence

$$0 \to \mathscr{O}_X(3K - C) \to \mathscr{O}_X(3K) \to \mathscr{O}_C(3K) \to 0,$$

which induces a sequence

$$H^0(\mathscr{O}_X(3K)) \to H^0(\mathscr{O}_C(3K)) \to H^1(\mathscr{O}_X(3K-C)) \to H^1(\mathscr{O}_X(3K)).$$

Received by the editors March 13, 1991 and, in revised form, May 17, 1991.

1991 Mathematics Subject Classification. Primary 14J29; Secondary 14J60.

©1993 American Mathematical Society 0002-9939/93 \$1.00 + \$.25 per page

TOHRU NAKASHIMA

Since $H^1(\mathscr{O}_X(3K)) \cong H^1(\mathscr{O}_X(-2K)) = 0$ by [E, Theorem 1.7], our assumption implies that the map $H^0(\mathscr{O}_X(3K)) \to H^0(\mathscr{O}_C(3K))$ is not surjective. Thus we have $H^1(\mathscr{O}_X(3K - C)) \cong H^1(\mathscr{O}_X(C - 2K)) \neq 0$. Then there is a nonsplit extension

$$0 \to \mathscr{O}_X \to E \to \mathscr{O}_X(2K - C) \to 0$$

where E is a rank 2 vector bundle. E satisfies the inequality $c_1(E)^2 = (2K - C)^2 = 4 > 0 = 4c_2(E)$.

By Theorem 1 in [SB], there exists a Frobenius map $F^e: X \to X$ such that the pullback $(F^e)^*E$ is unstable in the sense of Bogomolov. Therefore, we obtain an exact sequence

$$0 \to \mathscr{O}_X(p^e(2K-C)-\Delta) \to (F^e)^*E \to \mathscr{I}_A \otimes \mathscr{O}_X(\Delta) \to 0.$$

Here Δ is an effective divisor such that $p^e(2K - C) - 2\Delta$ is contained in the positive cone of X and A is a 0-dimensional subscheme (cf. [SB]). We shall consider the cases e = 0 and e > 0 separately and derive a contradiction.

(i) The case e = 0. First we note, in this case, $\Delta \neq 0$. For, if $\Delta = 0$ then deg A = 0 and our original sequence must split, which is a contradiction. Since $0 < (2K - C - 2\Delta) \cdot K = 2 - 2\Delta \cdot K$, we have $\Delta \cdot K = 0$. Thus Δ is a sum of (-2) curves and $\Delta^2 \leq -2$. On the other hand, deg $A = \Delta \cdot (C + \Delta - 2K) = \Delta \cdot (C + \Delta) \ge 0$, so $\Delta \cdot C \ge -\Delta^2 \ge 2$. Then

$$-2 \ge (C + \Delta)^2 = \Delta \cdot (C + \Delta) + \Delta \cdot C + C^2$$
$$\ge \Delta \cdot (C + \Delta) + 2 - 2 = \Delta \cdot (C + \Delta) \ge 0.$$

This is a contradiction.

(ii) The case e > 0. As in (i), $(p^e(2K - C) - 2\Delta) \cdot K > 0$ and thus we have $\Delta \cdot K \leq p^e - 1$. According to [SB], there exists a purely inseparable covering $\rho: Y \to X$. We denote by ω_Y its dualizing sheaf. Then we compute, as in [SB],

$$\omega_Y \cdot \rho^* K = 2(p^e - 1)K \cdot \Delta + p^e (K^2 - (p^e - 1)(2K - C) \cdot K)$$

$$\leq 2(p^e - 1)^2 + p^e (3 - 2p^e) = 2 - p^e < 0.$$

This implies that Y is ruled. Let q = q(X) be the irregularity of X. By Lemma 34 of [SB], $\chi(\mathscr{O}_Y) \leq 1 - q$. Furthermore, we have the inequality

$$\chi(\mathscr{O}_Y) \ge p^e \left\{ \chi(\mathscr{O}_X) + \frac{p^e - 1}{12} [(2p^e - 1)(2K - C)^2 - 3(2K - C) \cdot K] \right\}.$$

Since Corollary 1.8 of [E] gives $\chi(\mathscr{O}_X) \ge 1$, the above inequality shows $\chi(\mathscr{O}_Y) > 1$, which is impossible. \Box

Lemma 1.2. Suppose $K^2 = \chi(\mathscr{O}_X) = 1$ and $\operatorname{char}(k) = p > 2$. Then for an effective divisor D that is numerically equivalent to K (resp. 2K), we have $h^0(D) \leq 1$ (resp. $h^0(D) = 2$).

Proof. Assume D is numerically equivalent to K and $h^0(D) \ge 2$. Then a Clifford argument shows $h^0(2D) \ge 3$. Since K is nef and big, $h^2(2D) = h^0(K - 2D) = h^0(-K) = 0$. Moreover, by Corollary 22 of [SB] we have

 $h^{1}(2D) = h^{1}(-(2D - K)) = 0$. Therefore, the Riemann-Roch formula gives

$$h^{0}(2D) = \chi(2D) = \frac{2D \cdot (2D - K)}{2} + \chi(\mathscr{O}_{\chi})$$
$$= \frac{2K \cdot (2K - K)}{2} + 1 = 2.$$

Thus we have a contradiction. The second claim can be proved similarly. \Box

Lemma 1.3. Let X be as before. If D is an effective divisor such that $h^0(D) \ge 2$ then $K \cdot D \ge 2$.

Proof. It is clear that $K \cdot D > 0$. Suppose that $K \cdot D = 1$. Then $K \cdot (K - D) = 0$, and hence D is numerically equivalent to K or $(K - D)^2 < 0$. Since the former case is impossible by Lemma 1.2, we have $(K - D)^2 = 1 - 2KD + D^2 < 0$ and the result follows easily from this. \Box

Using the preceding results we shall prove the theorem. Let |3K| = |M| + F where M is the moving part and F the fixed part. Then $3 = 3K^2 = M \cdot K + F \cdot K$. By Proposition 1.1, we have $F \cdot K > 0$ and by Lemma 1.3, $M \cdot K \ge 2$. Thus $F \cdot K = 1$ and $M \cdot K = 2$, and hence $K \cdot (M - 2K) = 0$. Then we have $M^2 \le 3$ since Lemma 1.2 implies $(M - 2K)^2 < 0$. Let $\phi_{|M|}$ be the map associated to |M|. By assumption |M| is not composed with a pencil, so deg $\phi_{|M|} \cdot \text{deg } \phi_{|M|}(X) \le M^2 \le 3$. Since $h^0(3K) = 4$, $\phi_{|M|}(X)$ is a nondegenerate surface in \mathbf{P}^3 . Then deg $\phi_{|M|}(X) \ge 2$, and we obtain an obvious contradiction.

References

- [B] E. Bombieri, Canonical models of surfaces of general type, Inst. Hautes Études Sci. Publ. Math. 42 (1973), 171–220.
- [E] T. Ekedahl, Canonical models of surfaces of general type in positive characteristic, Inst. Hautes Études Sci. Publ. Math. 67 (1988), 97–144.
- [M] Y. Miyaoka, Tricanonical map of numerical Godeaux surfaces, Invent. Math. 34 (1976), 99-111.
- [SB] N. I. Shepherd-Barron, Unstable vector bundles and linear systems on surfaces in characteristic p, preprint.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KYOTO UNIVERSITY, KYOTO, 606 JAPAN Current address: Department of Mathematics, Faculty of Science, Tokyo Metropolitan University, Minami-Ohsawa 1-1, Hachioji-shi, Tokyo, 192-03 Japan