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NONUNIQUENESS FOR THE RADON TRANSFORM

D. H. ARMITAGE AND M. GOLDSTEIN

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ABSTRACT. There exists a nonconstant harmonic function h on $\mathbb{R}^N$, where $N \geq 2$, such that $\int_P |h| < +\infty$ and $\int_P h = 0$ for every $(N-1)$-dimensional hyperplane P.

Let f be a real- or complex-valued function on $\mathbb{R}^N$ $(N \geq 2)$, and suppose that f is integrable on each $(N-1)$-dimensional hyperplane P in $\mathbb{R}^N$. The Radon transform $\widehat{f}$ of f is defined on the set $\mathbb{P}^N$ of all such hyperplanes by $\widehat{f}(P) = \int_P f\,d\lambda$, where λ denotes $(N-1)$-dimensional Lebesgue measure on P. We refer to Helgason [4] for the general theory of the Radon transform and its applications.

There are several proofs that if f is continuous and integrable on $\mathbb{R}^N$ and $\widehat{f} \equiv 0$ on $\mathbb{P}^N$, then $f \equiv 0$ on $\mathbb{R}^N$ (see Zalcman [5] for references); the simplest proof proceeds by showing that, under the stated hypotheses, the Fourier transform of f vanishes identically. With $N = 2$, at least, the hypothesis that f is integrable on $\mathbb{R}^N$ cannot be removed. Indeed, identifying $\mathbb{R}^2$ with $\mathbb{C}$, Zalcman [5, §5] showed that there exists a nonconstant entire function ϕ such that $\hat{\phi} \equiv 0$ on $\mathbb{P}^2$. The real part h of ϕ provides an example of a nonconstant harmonic function on $\mathbb{R}^2$ such that $\widehat{h} \equiv 0$ on $\mathbb{P}^2$. Zalcman's proof depends on an approximation theorem of Arakelian [1, p. 1189] for holomorphic functions and has no obvious generalization to $\mathbb{R}^N$ $(N \geq 3)$. Here we use a recent theorem [3, Theorem 1.1] on harmonic approximation to prove the following result.

Theorem. *There exists a nonconstant harmonic function h on $\mathbb{R}^N$ $(N \geq 2)$ such that $\widehat{h} \equiv 0$ on $\mathbb{P}^N$.*

To the best of our knowledge, it has not hitherto been decided whether there exists even a nonconstant continuous function f on $\mathbb{R}^N$ $(N \geq 3)$ for which $\widehat{f} \equiv 0$ on $\mathbb{P}^N$.

We denote a typical point of $\mathbb{R}^N$ by $x = (x_1, \ldots, x_N)$ and write

$$\langle x, y\rangle = x_1y_1 + \cdots + x_Ny_N, \quad \|x\| = \sqrt{\langle x, x\rangle} \qquad (x, y \in \mathbb{R}^N).$$

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Let S denote the sphere $\{y \in \mathbb{R}^N : \|y\| = 1\}$. If $y \in S$ and $t \in \mathbb{R}$, we write

$$P(y, t) = \{x \in \mathbb{R}^N : \langle x, y \rangle = t\};$$

if $-\infty \le a < b \le +\infty$, we put

$$Q(y, a, b) = \bigcup_{a<t<b} P(y, t).$$

Thus $P(y, t)$ is an $(N-1)$-dimensional hyperplane and $Q(y, a, b)$ is isometric to $\mathbb{R}^{N-1} \times (a, b)$. The point at infinity of $\mathbb{R}^N$ is denoted by $\mathscr{A}$, and we understand $\mathbb{R}^N \cup \{\mathscr{A}\}$ to be equipped with the Aleksandroff one-point compactification topology.

To prove the theorem, we need a nonempty subset E of $\mathbb{R}^N$ with the following properties:

(i) E is open in $\mathbb{R}^N$;
(ii) $E \cup \{\mathscr{A}\}$ is connected and locally connected in the topology of $\mathbb{R}^N \cup \{\mathscr{A}\}$;
(iii) if $y \in S$ and $0 < a < +\infty$, then $E \cap Q(y, -a, a)$ is bounded;
(iv) if $y \in S$ then there exists a positive number T, depending on y, such that at least one of the sets $E \cap Q(y, -\infty, -T)$ and $E \cap Q(y, T, +\infty)$ is empty.

An example of such a set E is as follows. Let $I = [0, +\infty)$, define $\psi : I \to \mathbb{R}^N$ by $\psi(\xi) = (\xi, \xi^2, \dots, \xi^N)$, and put

$$E = \left\{x \in \mathbb{R}^N : \inf_{\xi \in I} \|x - \psi(\xi)\| < 1\right\}. \tag{1}$$

We owe this example to a remark of Dr. T. B. M. McMaster; it replaces a more complicated example of ours. It is clear that the set E defined by (1) has properties (i) and (ii); we verify at the end of this note that it also has properties (iii) and (iv).

Now fix a point z of a set E satisfying (i)–(iv) and define closed subsets of $\mathbb{R}^N$ by

$$F_1 = \mathbb{R}^N \setminus E, \qquad F_2 = \{z\}, \qquad F = F_1 \cup F_2.$$

Clearly F is unbounded. Let ω_1 and ω_2 be disjoint open subsets of $\mathbb{R}^N$ containing F_1 and F_2, respectively, and define a function u to be equal to 0 on ω_1 and equal to 1 on ω_2. Then u is harmonic on the open neighbourhood $\omega_1 \cup \omega_2$ of F. Also, properties (i) and (ii) hold with $\mathbb{R}^N \setminus F = E \setminus \{z\}$ in place of E. It follows from [3, Theorem 1.1] that there exists a harmonic function h on $\mathbb{R}^N$ such that

$$|h(x) - u(x)| < (1 + \|x\|)^{-N-1} \qquad (x \in F). \tag{2}$$

In particular, $|h(z) - 1| < 1$ and $\lim_{x \to \mathscr{A},\, x \in F} h(x) = 0$, so that h is nonconstant.

Let y be a point of S. It suffices to show that h is integrable on $P(y, t)$ and $\widehat{h}(P(y, t)) = 0$ for all real t. Suppose that $0 < a < +\infty$. By property (iii), we have for some positive number r

$$Q(y, -a, a) \setminus F \subset \{x \in \mathbb{R}^N : \|x\| < r\} = B_N(r), \quad \text{say}.$$

From this remark and (2) we obtain that when $|t| < a$

$$\int_{P(y,t)} |h|\,d\lambda \le \sup_{B_N(r)} |h| \int_{P(y,t)\setminus F} d\lambda + \int_{P(y,t)\cap F} (1+\|x\|)^{-N-1}\,d\lambda(x)$$
$$\le V(r) \sup_{B_N(r)} |h| + \int_{P(y,0)} (1+\|x\|)^{-N-1}\,d\lambda(x),$$

where $V(r)$ is the $(N-1)$-dimensional volume of $B_{N-1}(r)$. Thus the function $t \to \int_{P(y,t)} |h|\,d\lambda$ is locally bounded on $\mathbb{R}$. Now, using a rotation of axes, we find from known results (see, e.g., [2, Theorem 2]) that if s is subharmonic on $\mathbb{R}^N$ and the function $t \to \int_{P(y,t)} |s|\,d\lambda$ is locally bounded on $\mathbb{R}$, then the hyperplane mean $\hat{s}(P(y,t))$ is a convex function of t on $\mathbb{R}$. Applying this result with $s = h$ and with $s = -h$, we obtain that $\widehat{h}(P(y,t))$ is a linear function (i.e., a polynomial of degree at most 1) of t. By property (iv), there exists a positive number T such that $P(y,t) \subset F$ either for all $t > T$ or for all $t < -T$. Also, when $P(y,t) \subset F$ we obtain from (2) that

$$|\widehat{h}(P(y,t))| < \int_{P(y,t)} (1+\|x\|)^{-N-1}\,d\lambda(x)$$
$$= \int_{P(y,0)} (1+\sqrt{(\|x\|^2+t^2)})^{-N-1}\,d\lambda(x)$$
$$< (1+|t|)^{-1} \int_{P(y,0)} (1+\|x\|)^{-N}\,d\lambda(x),$$

so that $\widehat{h}(P(y,t)) \to 0$ either as $t \to +\infty$ or as $t \to -\infty$. Since $\widehat{h}(P(y,t))$ is a linear function of t, it now follows that $\widehat{h}(P(y,t)) = 0$ for all real t.

It remains to verify that the set E given by (1) has properties (iii) and (iv). Fix a point y of S and define $\eta: I \to \mathbb{R}$ by

$$\eta(\xi) = \sum_{j=1}^{N} y_j \xi^j.$$

Note that $|\eta(\xi)| \to +\infty$ as $\xi \to +\infty$ and that η is either bounded above or bounded below on I. For each point x of E, there exist a number ξ_x in I and a point x' of $B_N(1)$ such that $x = \psi(\xi_x) + x'$. Clearly $\xi_x \to +\infty$ as $x \to \mathscr{A}$ $(x \in E)$. We have

$$\langle x, y\rangle = \langle \psi(\xi_x), y\rangle + \langle x', y\rangle$$
$$= \eta(\xi_x) + O(1) \qquad (x \to \mathscr{A},\ x \in E).$$

It follows that if $0 < a < +\infty$ then $\{x \in E : |\langle x, y\rangle| < a\}$ is bounded, so that (iii) holds. It also follows that there exists a positive number T such that either $\langle x, y\rangle < T$ for all x in E or $\langle x, y\rangle > -T$ for all x in E, so that (iv) holds.

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Department of Pure Mathematics, The Queen's University of Belfast, Belfast BT7 1NN, Northern Ireland
E-mail address: d.armitage@uk.ac.qub.v2

Department of Mathematics, Arizona State University, Tempe, Arizona 85287