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ON THE JACOBIAN CONJECTURE

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ABSTRACT. We show that the Jacobian conjecture can be reduced to a weaker conjecture in which all fibers of coordinate functions are irreducible.

1. INTRODUCTION

Let (x, y) be a coordinate system in $\mathbf{C}^2$. The Jacobian Conjecture says that a polynomial mapping $(p, q)\colon \mathbf{C}^2 \to \mathbf{C}^2$ whose Jacobian $J(p, q) = \partial p/\partial x \cdot \partial q/\partial y - \partial p/\partial y \cdot \partial q/\partial x$ is equal to 1 is invertible. This conjecture first appeared in [K], and one can read a nice survey of the results that concentrated around the conjecture in [BCW]. Recall that a polynomial fiber is reducible if it is the union of more than one algebraic curve. Under the assumptions of the Jacobian conjecture, the system $\partial p/\partial x = \partial p/\partial y = 0$ has no solution. From this it follows that different components of a reducible fiber of p do not intersect and the polynomial p has no multiple fiber. Nontrivial polynomials of this kind with reducible fibers exist and $x(xy+1)$ is the simplest example. We shall formulate a new problem.

Weak Jacobian Conjecture. *Let $(p, q)\colon \mathbf{C}^2 \to \mathbf{C}^2$ be a polynomial mapping with $J(p, q) \equiv 1$. Suppose that for every $c \in \mathbf{C}$ the fiber $\{(x, y) | p(x, y) = c\}$ is irreducible. Then the mapping (p, q) is invertible.*

In other words, the additional condition on the polynomial fibers means that for every $c \in \mathbf{C}$ the polynomial $p(x, y) - c$ is prime. Our main result is the following

Theorem. *If the Weak Jacobian Conjecture is true then the Jacobian Conjecture is true.*

In order to prove this fact for each couple of polynomials (p, q) with $J(p, q) \equiv 1$ we shall find a polynomial automorphism $\alpha = (\alpha_1, \alpha_2)\colon \mathbf{C}^2 \to \mathbf{C}^2$ such that $p_1 = \alpha_1(p, q)$ does not have reducible fibers. We would like to note also that in order to prove the Weak Jacobian Conjecture one may try to show that a polynomial whose fibers are irreducible and different from $\mathbf{C}$ must have a fiber with a singular point. The last fact holds if this polynomial is good at infinity [NR]. Application of the current theorem enables us to simplify the proof of

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the results in [S1–S3], since the existence of reducible fibers makes the original proof more complicated. Recall an equivalent formulation of the Jacobian Conjecture: if $J(p, q) \equiv 1$ then the ring $\mathbf{C}[p, q]$ generated by p and q coincides with the polynomial ring $\mathbf{C}[x, y]$.

The following fact is the two-dimensional case of a theorem in [F] (it is a direct corollary of [W] as well).

Theorem A. *If $J(p, q) \equiv 1$ and there exists a polynomial h such that $\mathbf{C}[p, q, h] = \mathbf{C}[x, y]$, then $\mathbf{C}[p, q] = \mathbf{C}[x, y]$.*

Combination of our main result and Theorem 3 from [R] gives another proof of Theorem A.

2. Proof of Theorem

Let $P: M \to \mathbf{C}^2$ be a regular mapping from a smooth connected algebraic surface M to $\mathbf{C}^2$. Suppose that the dimension of the inverse image $P^{-1}(w)$ of every point $w \in \mathbf{C}^2$ is equal to 0. Then there exists an affine algebraic curve $\Gamma \subset \mathbf{C}^2$ such that for every $w \in \mathbf{C}^2 - \Gamma$ the inverse image $P^{-1}(w)$ consists exactly of n points (these points are not multiple). We shall say that Γ is the branch curve of P.

Definition. We shall say that an irreducible curve G is "Γ proper," if the embedding $i: G-\Gamma \to \mathbf{C}^2-\Gamma$ generates an epimorphism $i_*: \pi_1(G-\Gamma) \to \pi_2(\mathbf{C}^2-\Gamma)$ from the fundamental group $\pi_1(G-\Gamma)$ to the fundamental group $\pi_1(\mathbf{C}^2-\Gamma)$.

Lemma 1. *Let G be a smooth irreducible affine algebraic Γ proper curve. Then the inverse image $G^1 = P^{-1}(G)$ is an irreducible curve in M.*

Proof. Let $w \in \mathbf{C}^2-\Gamma$ and let $P^{-1}(w) = \{z_1, \dots, z_n\}$. Since M is connected, there is a path $s_{ij}: [0, 1] \to M - P^{-1}(\Gamma)$ for which $s_{ij}(0) = z_i$ and $s_{ij}(1) = z_j$. Put $\tau_{ij} = P \circ s_{ij}$. Then τ_{ij} is a loop and $\tau_{ij}(0) = \tau_{ij}(1) = w$. This loop generates an element $[\tau_{ij}]$ of the fundamental group $\pi_1(\mathbf{C}^2 - \Gamma, w)$. Let τ'_{ij} be another loop that generates $[\tau_{ij}]$. By the homotopy lifting theorem [FR, Chapter 4] $P^{-1}(\tau'_{ij})$ contains a path that connects the points z_i and z_j. Let S^1 be the set of singular points of G^1. Then the curve G^1 is irreducible if and only if the set $G^1 - S^1$ is connected. Suppose that $w \in A = G - (\Gamma \cup P(S^1))$. Since $i_*: \pi_1(G - \Gamma, w) \to \pi_1(\mathbf{C}^2 - \Gamma, w)$ is an epimorphism, we can choose all the loops $\{\tau_{ij}\}$ so that $\tau_{ij} \subset G - \Gamma$. Moreover, by perturbing these loops, we may suppose that $\tau_{ij} \subset A$. In order to prove that $G^1 - S^1$ is connected it is enough to present a path that connects the point z_1 with an arbitrary point $z' \in G^1 - S^1$. Let $w' = P(z')$. Since A is connected, there is a path $\tau: [0, 1] \to A$ with $\tau(0) = w$ and $\tau(1) = w'$. Then the inverse image $P^{-1}(\tau)$ contains a path that connects z' with z_j for a certain j. But z_1 and z_j are connected by s_{1j}. This implies the desired conclusion. □

The following fact is obvious.

Lemma 2. *Let $\{\gamma_i^t: [0, 1] \to \mathbf{C}^2 - \Gamma\}$ be a family of loops in $\mathbf{C}^2 - \Gamma$ that depend* [redacted]

Lemma 3. *Let $f(c, u, v)$ be a polynomial in $\mathbf{C}^3$. Let $F_0 = \{(u, v) \in \mathbf{C}^2 | f(0, u, v) = 0\}$. Assume F_0 contains a smooth irreducible component E so that either $\partial f/\partial u|_E$ or $\partial f/\partial v|_E$ does not equal 0 identically. If the curve E is Γ proper, then there exists a neighborhood U of the origin in $\mathbf{C}$ such that for every $c \in U$ the curve $F_c = \{(u, v) \in \mathbf{C}^2 | f(c, u, v) = 0\}$ contains a component that is Γ proper.*

Proof. Choose loops $\gamma_i^0\colon [0, 1] \to E - \Gamma$ with $w_0 = \gamma_i^0(0)$ for every i such that these loops generate the group $\pi_1(\mathbf{C}^2 - \Gamma, w_0)$. Consider $K_0 = \bigcup_i \gamma_i^0$. Since the manifold $\mathbf{C}^2 - \Gamma$ is an algebraic variety, its fundamental group is finitely generated (which follows from $[Z]$ as well). Thus we may suppose that T is compact. Without loss of generality, consider only the case when $\partial f/\partial v$ is not identically zero on E. Since E is smooth, one can perturb γ_i^0 a little so that $\partial f/\partial v$ is different from zero at every point of T. Let V_0 be a sufficiently small neighborhood of T in $\mathbf{C}^2 - \Gamma$. Choose a sufficiently small neighborhood U of the origin in $\mathbf{C}$, and put $V^1 = \{(c, u, v) \in \mathbf{C}^3 | (u, v) \in V_0, \ c \in U\}$. One may suppose that $\partial f/\partial v$ is different from 0 in V^1. Then there exists the function ν_v for which $\partial f/\partial c + \partial f/\partial v \cdot \nu_v \equiv 0$ on V^1. Hence the vector field $(1, 0, \nu_v)$ is tangent to the surface $H = \{(c, u, v) | f(c, u, v) = 0\} \subset \mathbf{C}^3$. Let φ_t be the phase flow associated with this field. Put $K_0 = \bigcup_i \gamma_i^0$. Then $K_t = \varphi_t(K_0) \subset F_t$ for every arbitrarily small t. The set K_t is the union of loops $\gamma_i^t = \varphi_t(\gamma_i^0)$, and we are under the assumptions of Lemma 2. For each fixed t these loops belong to the same component of the curve F_t, since they have the common point $\phi_t(\omega_0)$. This concludes the proof of Lemma 3. □

To each complex line $\{(u, v) \in \mathbf{C}^2 | au + bv + c = 0\}$ in $\mathbf{C}^2$ one may assign the point $(a, b, c) \in \mathbf{CP}^2$. Thus we can consider the set of complex lines in $\mathbf{C}^2$ as $W = \mathbf{CP}^2 - (0, 0, 1)$ (the point $(0, 0, 1)$ has to be deleted since a and b cannot equal 0 simultaneously). Let the mapping $\rho\colon W \to \mathbf{CP}^1$ be given by the formula $(a, b, c) \to (a, b)$. Suppose that $P\colon M \to \mathbf{C}^2$ and Γ are the same as above. Let $\Gamma^\star$ be the dual curve, i.e., $\Gamma^\star$ is the closure in W of the set of points that correspond to tangent lines to Γ at the smooth part of Γ. We shall denote the set of singular points of Γ of S. For every $w \in \mathbf{C}^2$ we put $K(w) = \{\ell \in W | w \in \ell\}$ (here we consider ℓ as both a point in W and as a line in $\mathbf{C}^2$) and $K = \bigcup_{w \in S} K(w) \subset W$. Let the curve Γ be given by an equation $h(u, v) = 0$ and let $\Lambda = \{\lambda_i = (a_i, b_i)\} \subset \mathbf{CP}^1$ be the set of roots of the leading homogeneous part of h. For each $\lambda \in \mathbf{CP}^1$ we put $L(\lambda) = \rho^{-1}(\lambda)$ and $L = \bigcup_{\lambda \in \Lambda} L(\lambda)$. Suppose that $P = (p, q)$ is the coordinate representation of P. Let X be the closure (in the Euclidean topology of W) of the set of points $\{(a, b, c)\}$ such that the curve $\{z \in M | ap(z) + bq(z) + c = 0\}$ is reducible.

Lemma 4. *The set X is contained in $\Gamma^\star \cup K \cup L$.*

Proof. We may require that the polynomial h does not have repeated factors. Put $m = \deg h$. Recall that a line ℓ is in general position relative to Γ, if the set $\ell \cap \Gamma$ consists of m different points. By Lefschetz's theorem, for every line $\ell \subset \mathbf{C}^2$ in a general position the embedding $i\colon \ell \hookrightarrow \mathbf{C}^2$ induces an epimorphism $i_\star\colon \pi_1(\ell - \Gamma) \to \pi_1(\mathbf{C}^2 - \Gamma)$ (see [Z] or [A]). Thus, by Lemma 1, it is enough to check that if a line $\ell \notin \Gamma^\star \cup K \cup L$ then ℓ is in a general position. Let $\mathbf{C}^2 \hookrightarrow \mathbf{CP}^2$ be a natural embedding, $E = \mathbf{CP}^2 - \mathbf{C}^2$, $\overline{\Gamma}$ be the closure of Γ in $\mathbf{CP}^2$, and $\overline{\ell}$ be the closure of ℓ in $\mathbf{CP}^2$. Then the set $\overline{\ell} \cap \overline{\Gamma}$ consists of m points for a line $\overline{\ell}$ in a general position. Note that if $\ell \notin L$ then $E \cap \overline{\ell} \cap \overline{\Gamma} = \varnothing$, i.e., $\overline{\ell} \cap \overline{\Gamma} = \ell \cap \Gamma \subset \mathbf{C}^2$. If $\ell \notin \Gamma^\star \cup K$ as well, then ℓ meets Γ transversally. Thus each $\ell \notin \Gamma^\star \cup K \cup L$ meets Γ transversally at m different points in $\mathbf{C}^2$. □

Note that if $\rho(X)$ is a finite set, then for every b such that $(1, b) \notin \rho(X)$ each fiber of the regular function $p(z) + bq(z)$ is irreducible. Since $\rho(L) = \Lambda$ is a finite set, the set $\rho(X)$ is finite in $X \cap (K \cup \Gamma^\star)$ is finite, or in other words, $\dim X \cap (K \cup \Gamma^\star) = 0$.

Lemma 5. *The set X is algebraic.*

Proof. Let D be an irreducible component of $\Gamma^\star \cup K \cup L$. It suffices to prove that the set $D \cap X$ is algebraic. Consider the algebraic variety $B = \{(x, \ell) | x \in P^{-1}(\ell), \ell \in D\} \subset M \times W$. Let $\tau \colon B \to D$ be the natural projection, and let $n(c)$ be the number of irreducible components of a fiber $Q_c = \{x | (x, c) \in B\}$ of this mapping. We shall show later that the function $n(c)$ on D coincides with a constant n outside a finite set $C \subset D$. (Actually, a stronger fact holds. It is well-known that outside a finite subset of D all fibers are diffeomorphic. For our purpose it suffices to check the simpler assertion.) If $n > 1$ then $X \supset D - C$, i.e., $X \cap D = D$. Otherwise $X \cap D \subset C$ and $X \cap D$ is again an algebraic variety. Now we have to show that the set C is finite. Let B' be the smooth part of B. The algebraic variety $A = B - B'$ is the union of a finite number of irreducible algebraic curves and points. Now we define the set $D_0 \subset D$ such that for $c \in D_0$ an irreducible component of the set A belongs to the fiber Q_c. Hence the set D_0 is finite. Note that the number of components of $Q'_c = Q_c \cap B'$ coincides with the number of components of Q_c, when $c \notin D_0$. Consider a smooth compact algebraic curve $\overline{D} \supset D - D_0$. Put $\phi = \tau|_{B'}$. Standard results from the theory of resolution of singularities yield the existence of a smooth compact algebraic variety $\overline{B} \supset B'$ for which there exists an extension $\overline{\phi} \colon \overline{B} \to \overline{D}$ of the mapping ϕ. Let B_1 be the subset of points in $\overline{B}$ for which the mapping $\overline{\phi}$ is not smooth. Then $D_1 = \overline{\phi}(B_1)$ is a finite algebraic subvariety of $\overline{D}$ [M, Proposition 3.7]. The proper mapping $\overline{\phi} \colon \overline{B} - \overline{\phi}^{-1}(D_1) \to \overline{D} - D_1$ is smooth at every point and, therefore, it is a smooth fibration [MK, Chapter 1, Theorem 4.1]. In particular, the number of irreducible components of the curve $\overline{Q}_c = \overline{\phi}^{-1}(c)$ is constant when $c \notin D_1$. The set $A' = \overline{B} - B'$ is an algebraic curve. Let D_2 be the subset of points $c \in \overline{D}$ such that $\overline{\phi}(E) = c$ for a certain irreducible component E of the curve A'. Since the number of irreducible components of the curve A' is finite, the set D_2 is finite. Put $C' = D_1 \cup D_2$. If $c \notin C'$ then the set $\overline{Q}_c - Q'_c$ is finite. Thus the curves $\overline{Q}_c$ and Q'_c have the same number of irreducible components. The algebraic set $\overline{D} - (D - D_0)$ is finite. □

Lemma 6. *Let ℓ be a Γ proper line in $K(w)$ for some point w. The dimension of $X \cap K(w) = 0$.*

Proof. One may suppose that w coincides with the origin and that ℓ is $\{u = 0\}$. Let $\ell_c = \{(u, v) | u + cv = 0\}$. By Lemma 3, we have the epimorphism $i^c_\star \colon \pi_1(\ell_c - \Gamma) \to \pi_1(\mathbf{C}^2 - \Gamma)$, when $|c|$ is sufficiently small. By Lemma 1 this implies that $P^{-1}(\ell_c)$ is irreducible. Since $X \cap K(w)$ is an algebraic variety, there can be only finite number of complex numbers $\{c_k\}$ such that the line $\ell_{c_k} \in K$. Thus $\dim X \cap K(w) = 0$. $\square$

Note that if Γ is reducible then Γ^* is the union of the duals of its reducible components.

Lemma 7. *Let Γ_i be an irreducible component of Γ, let Γ_i^* be the dual curve for Γ_i, and let l be a tangent line to Γ_i at a regular point w_i. If ℓ is Γ proper then* $\dim X \cap \Gamma_i^* = 0$.

Proof. We may again suppose that w is the origin and ℓ is $\{u = 0\}$. The curve Γ_i is locally given by the equation $u = u(v) = av^n + v^{n+1}h(v)$, where $a \neq 0$ and $n \geq 2$. For every point $(u(c), c) \in \Gamma_i$ when $|c|$ is sufficiently small, we have the following equation for the tangent line to Γ_i at the point

$$u - nac^{n-1}v + (n-1)ac^n + v \cdot O(|c|^n) + O(|c|^{n+1}) = 0.$$

Since $X \cap \Gamma_i^*$ is an algebraic variety, application of Lemma 3 again provides the desired conclusion. $\square$

Let $A_g \colon \mathbf{C}^2 \to \mathbf{C}^2$ be a polynomial automorphism given by the formula

$$A_g(u, v) = (u + g(v), v),$$

where $g \in \mathbf{C}[v]$ is a polynomial in one variable. Put $P_g = A_g \circ P$, and let the symbols Γ_g, Γ_{ig}, Γ_g^*, Γ_{ig}^*, X_g, $K_g(w)$, K_g, L_g, S_g have the same meaning for the mapping P_g as the symbols Γ, Γ_i, Γ^*, Γ_i^*, X, $K(w)$, K, L, S for P.

Lemma 8. *Let the line $\{v = 1\}$ be Γ proper. Then there exists a polynomial $g \in \mathbf{C}[v]$ satisfying*

(1) *for every irreducible component $\Gamma_{ig} \subset \Gamma_g$ there is a regular point $w_i' \in \Gamma_{ig}$ such that the tangent line ℓ_i^1 to Γ_{ig} at w_i' is Γ_g proper;*
(2) *for every $z_j' \in S_g$ there exists a line $\ell_j^2 \in K_g(z_j')$ that is Γ_g proper.*

Proof. Choose $w_i \in \Gamma_i$ so that the tangent line to Γ_i at w_i is given by the equation $u + b_i^1 v + d_i^1 = 0$, where $b_i^1, d_i^1 \in \mathbf{C}$. Let V be the set of the v-coordinates of the points of the set $S \cup \{w_i\}$. Choose a polynomial $h \in \mathbf{C}[v]$ so that for every $v_0 \in V$ we have

$$h(v_0) = \frac{dh}{dv}(v_0) = 0. \tag{1}$$

Put $g(v) = c(v-1) \cdot h(v)$, where c is an arbitrarily large number. Suppose that a line $\ell_j^2 = \{u + b_j^2 v + d_j^2 = 0\}$ belongs to $K(z_j)$, where $z_j \in S$. Choose a line ℓ given by the equation $\{au + bv + d = 0\}$. Then the curve $\ell_g = A_g^{-1}(\ell)$ coincides with

$$au + ag(v) + bv + d = 0. \tag{2}$$

Vice versa: for each curve ℓ_g given by the equation (2) the curve $A_g(\ell_g)$ is a complex line. For $k = 1, 2$ put $\ell_g^{ki} = \{(u, v) | u + g(v) + b_i^k v + d_i^k = u + b_i^k v + d_i^k + c(v-1)h(v) = 0\}$. By (1), the curve ℓ_g^{1i} is tangent to Γ_i at w_i.

Hence $A_g(\ell_g^{1i})$ is tangent to $A_g(\Gamma_i)$ at $w_i' = A_g(w_i)$. Note that $A_g(\Gamma_i) = \Gamma_{ig}$, $A_g(\Gamma) = \Gamma_g$, and the equation of the curve ℓ_g^{ki} can be written in the form

$$c^{-1}(u + b_i^k v + d_i^k) + (v - 1)h(v) = 0.$$

Recall that the line $\{v = 1\}$ is Γ proper. Application of Lemma 3 shows that ℓ_g^{ki} is Γ proper. Hence each line $l_i^k = A_g(\ell_g^{ki})$ is Γ_g proper. It remains to note that $S_g = A_g(S)$. This concludes the proof of Lemma 8. □

Let $P = (p, q)\colon \mathbf{C}^2 \to \mathbf{C}^2$ have the Jacobian $J(p, q) \equiv 1$. Using an affine automorphism of $\mathbf{C}^2$ if necessary, one may suppose that the line $\{v = 1\}$ is Γ proper. Let A_g be the same as in Lemma 8. Then P is invertible iff $P_g = A_g \circ P$ is invertible. Keeping the previous notation, one may assert (by Lemma 7) that $\dim X_g \cap (K_g \cup \Gamma_g^\star) = 0$. Thus $\rho(X_g)$ is a finite set. If a point $(1, b) \notin \rho(X_g)$ and $P_g = (p_1, q_1)$ is the coordinate representation of P_g, then the polynomial $p_1 + bq_1$ has irreducible fibers only. The theorem is proved.

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