PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 117, Number 1, January 1993

A NOTE ON THE NORMAL GENERATION OF AMPLE LINE BUNDLES ON AN ABELIAN SURFACE

AKIRA OHBUCHI

(Communicated by Louis J. Ratliff, Jr.)

ABSTRACT. Let L be an ample line bundle on an abelian surface A. We prove that the four conditions: (1) L is base point free, (2) L is fixed component free, (3) $L^{\otimes 2}$ is very ample, (4) $L^{\otimes 2}$ is normally generated, are equivalent if $(L^2) > 4$. Moreover we prove that $L^{\otimes 2}$ is not normally generated if $(L^2) = 4$.

1. INTRODUCTION

Let B be an abelian variety defined over an algebraically closed field, and let M be an ample line bundle on B. The theorem of Lefschetz says that $M^{\otimes n}$ is very ample if $n \geq 3$. If a polarized abelian variety (B, M) is isomorphic to $(B_1 \times B_2, \mathcal{O}(\Theta_1 \times B_2 + B_1 \times D_2))$ where Θ_1 is a principal polarization of B_1, $\dim(B_1) > 0$, and $\dim(B_2) \geq 0$, then $M^{\otimes 2}$ is not very ample. Therefore the Lefschetz theorem gives a best possible condition for the very ampleness of $M^{\otimes n}$. The above example, however, is the only example for which $M^{\otimes 2}$ is not very ample (see Ohbuchi [3]). The condition that $M^{\otimes n}$ is normally generated is also given (see Koizumi [1], Sasaki [5], Sekiguchi [6–8]) and $n = 3$ is best possible in this case too. As for $M^{\otimes 2}$, if M is base point free then $M^{\otimes 2}$ is normally generated (see Ohbuchi [4]). In this paper we consider the difference between this very ample condition and this normally generated condition. The result is the following:

Theorem. *Let* L *be an ample line bundle on an abelian surface* A *defined over an algebraically closed field with characteristic* 0. *If* $(L^2) > 4$ *then the following conditions are equivalent*:

(1) L *is fixed component free,*
(2) L *is base point free,*
(3) $L^{\otimes 2}$ *is very ample,*
(4) $L^{\otimes 2}$ *is normally generated.*

If $(L^2) \leq 4$ then the above conditions are not equivalent. If $(L^2) = 2$ then $L^{\otimes 2}$ is not very ample, and therefore is not normally generated. In the last part of this paper, we prove that $L^{\otimes 2}$ is not normally generated if $(L^2) = 4$.

Received by the editors February 7, 1991 and, in revised form, May 23, 1991.
1991 *Mathematics Subject Classification.* Primary 55R25.

2. Proof of the theorem

To prove the above theorem, we prepare several lemmas. Throughout this paper, we assume that L is an ample line bundle on an abelian surface A defined over an algebraically closed field.

Lemma 1. *Let B be an abelian variety and M be an ample line bundle on B. $M^{\otimes 2}$ is not very ample if and only if the polarized variety $(B, \underline{M})$ is isomorphic to $(B_1 \times B_2, \mathscr{O}(D_1 \times B_2 + B_1 \times D_2))$ where B_1 and B_2 are abelian varieties with $\dim B_1 > 0$ and D_1 and D_2 are ample divisors with $\dim \Gamma(B_1, \mathscr{O}(D_1)) = 1$.*

Proof. See Ohbuchi [3]. Q.E.D.

Lemma 2. *L is fixed component free if and only if $L^{\otimes 2}$ is very ample.*

Proof. We may assume that $(L^2) > 2$. We assume that L has a fixed component. Let F be a fixed part of a complete linear system $|L|$, and put $|L| = |M| + F$. If $(M^2) > 0$ then M is ample. Moreover, $\underline{L} \simeq \underline{M} \otimes \mathscr{O}(F)$ implies $(L^2) = ((M+F)^2)$. Therefore $(M \cdot F) = (F^2) = 0$. This is a contradiction because M is an ample line bundle. Hence $(M^2) = 0$. Moreover, we can see that $(F^2) = 0$. In fact, we assume that $(F^2) > 0$. In this case, if $x \in K(L) = \{x \in A,\ T_x^* L \simeq L\}$ then $x \in K(F) = \{x \in A,\ T_x^* F$ is linearly equivalent to $F\} = \{x \in A;\ T_x^* F = F\}$. By $(F^2) > 0$, the order of $K(F) = (\dim_k \Gamma(A, \mathscr{O}(F)))^2 = 1$. Hence $(L^2) = 2$. This is a contradiction. Therefore (A, L) is isomorphic to $(A_1 \times A_2, \mathscr{O}(D_1 \times A_2 + A_1 \times D_2))$ where $A_1 = A/K^\circ(M)$, $A_2 = A/K^\circ(F)$, $K^\circ(M) =$ the connected component of $K(M) \ni 0$, $K^\circ(F) =$ the connected component of $K(F) \ni 0$, and D_1, D_2 are divisors corresponding to M, F. As F is a fixed part, $\dim_k \Gamma(A_2, \mathscr{O}(D_2)) = 1$. Therefore, we obtain this lemma by Lemma 1. Q.E.D.

Lemma 3. *Let B be an abelian variety of dimension g, and let M be an ample line bundle whose base points are at most a finite set. If $(M^g) > (g!)^2$ then M is base point free.*

Proof. We assume that the set of all base points $Bs|M|$ is not empty. Let $p \in Bs|M|$. We consider the set $K(M) = \{x \in B;\ T_x^* M \simeq M\}$. By the definition of $K(M)$, $p + K(M) \subset Bs|M|$. Hence $((M^g)/g!)^2 \leq (M^g)$ because the order of $K(M)$ is $((M^g)/g!)^2$. Therefore this lemma is obtained. Q.E.D.

Lemma 4. *Let M be a symmetric ample line bundle on a g-dimensional abelian variety B. Let $\alpha, \beta \in \hat{B}$ ($=$ the dual abelian variety) and let $u \in \hat{B}$ be an element such that $2u = \alpha - \beta$. Then $\Gamma(B, \underline{M}^2 \otimes \mathscr{P}_\alpha) \otimes \Gamma(B, \underline{M}^2 \otimes \mathscr{P}_B) \to \Gamma(B, \underline{M}^4 \otimes \mathscr{P}_{\alpha+\beta})$ is surjective if and only if $\eta + u$ is not contained in $\phi_M(Bs|M|)$ for every $\eta \in \hat{B}[2] = \{x \in \hat{B};\ 2x = 0\}$ where $\mathscr{P}$ is a Poincaré bundle and $\phi_M : B \to \hat{B}$ is a dual map.*

Proof. See Ohbuchi [4]. Q.E.D.

Proof of the theorem. The implications $(2) \to (1)$ and $(4) \to (3)$ are obvious. By Lemma 2, the conditions (1) and (3) are equivalent. By Lemma 4, $(2) \to (4)$ is obtained. By Lemma 3, $(1) \to (2)$ is obtained if $(L^2) > 4$. Q.E.D.

If $(L^2) = 2$ then L is a principal polarization. Therefore $L^{\otimes 2}$ is not very ample, so $L^{\otimes 2}$ is not normally generated. If $(L^2) = 4$ then the above equivalence does not hold because L always has base points. But in this case, as for $L^{\otimes 2}$, we can show that $L^{\otimes 2}$ is never normally generated and that $L^{\otimes 2}$ is very ample provided that L is fixed component free. We check this result.

Lemma 5. *If $(L^2) = 4$ and L is fixed component free, then a general $C \in |L|$ is smooth.*

Proof. By Bertini's theorem (see Zariski [9]), a general member $C \in |L|$ is smooth at $p \in C - Bs|L|$. Let $p \in Bs|L|$. Since $p + K(L) \subset Bs|L|$ and the order of $Bs|L| \leq 4$, it follows that $Bs|L|$ is a 4-point set. For every $p \in Bs|L|$, the intersection multiplicity at $p = (L \cdot C)_p \geq 1$. Hence C is smooth at $p \in Bs|L|$ because $(C^2) = 4$. Q.E.D.

Lemma 6. *$L^{\otimes 2}$ is not normally generated.*

Proof. We may assume that L is fixed component free. We also assume that $Bs|L| \ni 0$. As $K(L) = Bs|L|$, $Bs|L|$ is contained in $A[2]$. Let C be a smooth member of $|L|$. Let $\iota\colon A \to A$ be a morphism defined by $\iota(x) = -x$. In this case we obtain that $\iota^*C = C$. Because the condition $\iota^*C \neq C$ implies that C and ι^*C have a same tangent direction at every $p \in Bs|L|$, it follows that $(d\iota)_p\colon T_p(A) \to T_p(A)$ is multiplication by -1. Therefore $(C \cdot \iota^*C)_p \geq 2$. This is a contradiction. Hence C is a symmetric divisor on A. By Lemma 4, we obtain this lemma. Q.E.D.

References

1. S. Koizumi, *Theta relations and projective normality of abelian varieties*, Amer. J. Math. **98** (1976), 868–889.
2. D. Mumford, *Abelian varieties*, Oxford Univ. Press, New York, 1970.
3. A. Ohbuchi, *Some remarks on ample line bundles on abelian varieties*, Manuscripta Math. **57** (1987), 225–238.
4. ——, *A note on the normal generation of ample line bundles on abelian varieties*, Proc. Japan Acad. Ser. A Math. Sci. **64** (1988), 119–120.
5. R. Sasaki, *Theta relations and their applications in abstract geometry*, Sci. Rep. Tokyo Kyoiku Daigaku **381** (1977), 138–160.
6. T. Sekiguchi, *On projective normality of abelian varieties*, J. Math. Soc. Japan **28** (1976), 307–322.
7. ——, *On projective normality of abelian varieties*. II, J. Math. Soc. Japan **29** (1977), 709–727.
8. ——, *On the cubics defining abelian varieties*, J. Math. Soc. Japan **30** (1977), 703–721.
9. O. Zariski, *Introduction to the problem of minimal models in the theory of algebraic surfaces*, Math. Soc. Japan (1958).

Department of Mathematics, Faculty of Education, Yamaguchi University, Yamaguchi-shi, Yamaguchi 753, Japan