PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 117, Number 1, January 1993

TOPOLOGICAL COMPLETIONS OF METRIZABLE SPACES

BEN FITZPATRICK, JR., GARY F. GRUENHAGE, AND JAMES W. OTT

(Communicated by Dennis Burke)

ABSTRACT. For a pair of metrizable spaces X and Y, we investigate conditions under which there is a dense embedding $h: X \to Z$, where Z is completely metrizable and $Z \backslash h(X)$ is homeomorphic to Y. In such a case, Z is called a topological completion of X and Y is called a completion remainder of X. In case X and Y are completely metrizable, we give necessary and sufficient conditions that Y be a completion remainder of X. We characterize the completion remainders of $\mathbf{R}$ and those of the rationals, $\mathbf{Q}$. We also characterize the remainders of $\mathbf{Q}(\kappa)$, a nonseparable analogue of $\mathbf{Q}$.

1. INTRODUCTION

Wilanski [4] asked whether there is a 3-point completion of the reals, i.e., is there a dense embedding h of $\mathbf{R}$ into a Polish space Z such that $|Z \backslash h(\mathbf{R})| = 3$? More generally, one might ask under what conditions on metrizable spaces X and Y does there exist a homeomorphism h of X into a completely metrizable space Z such that $h(X)$ is dense in Z and $Z \backslash h(X)$ is homeomorphic to Y. In such a case, Z is called a topological completion of X and Y is called a completion remainder of X. In case $h(X)$ is open in Z, Y is called a closed completion remainder of X. Throughout, if X is a space, $d(X)$, $w(X)$, and $e(X)$ denote, respectively, the density, weight, and extent of X and $ld_p(x)$ denotes the local density at p of X:

$$d(X) = \omega + \inf\{|M| : M \text{ is dense in } X\};$$
$$w(X) = \omega + \inf\{|\mathscr{B}| : \mathscr{B} \text{ is a basis for } X\};$$
$$e(X) = \omega + \sup\{|D| : D \text{ is a closed, discrete set in } X\};$$
$$ld_p(X) = \omega + \inf\{|D| : D \text{ is dense in some open } U \text{ in } X,\ p \in U\}.$$

For metrizable spaces, $d(X) = w(X) = e(X)$. This and other relevant properties are to be found in Engelking [1].

In §2 we characterize those pairs (X, Y) of completely metrizable spaces such that Y is a closed completion remainder of X and those pairs such that Y is a completion remainder of X. Section 3 provides a characterization of the completion remainders of $\mathbf{Q}$, the rationals, and gives both necessary and sufficient conditions (neither being necessary and sufficient) for a space to

Received by the editors August 7, 1990 and, in revised form, May 6, 1991.
1991 *Mathematics Subject Classification*. Primary 54E99.
Key words and phrases. Completely metrizable, topological completions, Polish spaces, σ-compact, σ-discrete, nowhere locally compact.

be a completion remainder of $\mathbf{P}$, the irrationals. Section 4 studies completion remainders of nonseparable analogs $\mathbf{Q}(\kappa)$ and $\mathbf{P}(\kappa)$ of $\mathbf{Q}$ and $\mathbf{P}$. We conclude in §5 with two open questions.

2. The completely metrizable case

It is easy to see that if X and Y are metrizable and Y is a completion remainder of X, then $d(Y) \leq d(X)$ and that if Y is nonempty then X is not compact. Whether or not $e(X)$ is achieved, that is, whether there exists a closed discrete set in X whose cardinality is $e(X)$, plays an important role. Note that, for metrizable spaces X, if $e(X)$ is not achieved then $e(X)$ has countable cofinality. We begin with an example.

Example 1. A completely metrizable space $Z = X \cup Y$, where $X \cap Y = \varnothing$, X is dense in Z, and there is no closed discrete set in X of cardinality $d(Y)$. Consider a hedgehog H, centered at a point $\mathscr{O}$, where $H = \bigcup_{i<\omega} H_i$ and each H_i has $\aleph_i$ spines of length 1. That is, $H_i = \{\mathscr{O}\} \cup \bigcup_{\lambda \in \Lambda_i}((0, 1] \times \{\lambda\})$, where Λ_i is an indexing set of cardinality $\aleph_i$. Take $\Lambda_i \cap \Lambda_j = \varnothing$ for $i \neq j$. Let $H_i' = \{\mathscr{O}\} \cup \bigcup_{\lambda \in \Lambda_i}((0, 2^{-i}] \times \{\lambda\})$; let $Z = \bigcup_{i<\omega} H_i'$; let $Y = \{\langle 2^{-i}, \lambda\rangle : i < \omega, \lambda \in \Lambda_i\}$; and let $X = Z \backslash Y$. Note that Y is a discrete space of cardinality $\aleph_\omega$ but every discrete subset of X of cardinality $\aleph_\omega$ has $\mathscr{O}$ as a limit point.

This example motivates the following useful lemma.

Lemma 1. *Suppose X is a metrizable space and $e(X)$ is not achieved. Then there exists a point x of X such that $ld_x(X) = e(X)$. Moreover, the set of all such points is compact.*

Proof. Assume that $e(X)$ is not achieved. Let $\alpha_{-1} = 0$. Let $\{\alpha_n : n < \omega]$ be an increasing sequence of cardinals whose sum is $e(X)$. Suppose $ld_p(X) < e(X)$ for all $p \in X$. There is a minimal, locally finite open cover $\mathscr{U}$ of X such that for all $U \in \mathscr{U}$, $d(U) < e(X)$. Then $|\mathscr{U}| < e(X)$. If there is a cardinal $\alpha < e(X)$ such that $d(U) < \alpha$ for all $U \in \mathscr{U}$, then $d(X) \leq \alpha \cdot |\mathscr{U}| < e(X)$, which is impossible. For each $n < \omega$ there is a $U_n \in \mathscr{U}$ such that $d(U_n) \geq \alpha_n$. Since $d(U_n) = d(\overline{U}_n) = e(\overline{U}_n)$, there is a closed discrete set D_n in $\overline{U}_n$ such that $|D_n| \geq \alpha_{n-1}$. Then $D = \bigcup_{n \in \omega} D_n$ is closed and discrete and has cardinality $e(X)$, a contradiction.

Next, assume that there is an infinite closed and discrete set $A = \{a_n : n < \omega\}$ such that $ld_a(X) = e(X)$ for every $a \in A$. There is a discrete collection $\{U_n : n < \omega\}$ of open sets screening A. For each $n < \omega$, there is a closed discrete set D_n in $\overline{U}_n$ of cardinality $\geq \alpha_{n-1}$. Then $D = \bigcup_{n<\omega} D_n$ is closed and discrete in X and has cardinality $e(X)$, which is impossible.

Theorem 1. *Let X and Y be completely metrizable spaces. Then Y is a closed completion remainder of X if and only if there is a closed discrete subset of X of cardinality $d(Y)$.*

Proof. Suppose first that Y is a closed completion remainder of X, $h\colon X \to Z$ is a dense embedding, Z is completely metrizable, $Z \backslash h(X)$ is closed in Z and is homeomorphic to Y. Since $d(Y) \leq d(X) = e(X)$, it follows that if $e(X)$ is achieved or if $d(Y) < e(X)$ then there is a closed discrete set in X of cardinality $d(Y)$. Assume then that $d(Y) = e(X)$ and that $e(X)$ is not achieved. The set K of all points of X at which X has local density $e(X)$ is compact. Thus $h(K)$ and $Z\backslash h(X)$ are disjoint closed sets in Z and can be enclosed in open sets U_X and U_Y with disjoint closures. Let

$$Z' = Z \cap \overline{U}_Y, \quad X' = h(X) \cap \overline{U}_Y, \quad Y' = (Z\backslash h(X)) \cap \overline{U}_Y.$$

Then Z' is completely metrizable, $Y' \simeq Y$, Y' is closed in Z', and X' is dense in Z'. Now, since $X' \cap U_X = \varnothing$, then for all $p \in X'$ we have $ld_p(X') < e(X) = e(Y) + e(X')$; therefore $e(X')$ is achieved. Let D be a closed discrete set in X' of cardinality $e(X)$. Then $h^{-1}(D)$ is closed and discrete in X.

Next, assume that X has a closed discrete subset of cardinality $d(Y) = \alpha$. For some (perhaps finite) cardinal β, Y has a dense subset K of cardinality β. We further assume that $X \cap Y = \varnothing$ (otherwise, take disjoint copies X' and Y'). Since α is infinite and $\beta \leq \alpha$, there is a discrete collection $\mathscr{H}$ of open sets in X of cardinality $\beta \cdot \omega$. Let H be an Axiom of Choice set for $\mathscr{H}$, and let T be the induced mapping from H onto $\mathscr{H}$. Now, $H = \bigcup_{n<\omega} H_n$, where $H_m \cap H_n = \varnothing$ for $m \neq n$, and $|H_n| = \beta$, $n < \omega$. For each n, let T_n denote a bijection from K onto H_n.

There exists a sequence $\{G'_n : n < \omega\}$ for X as in Moore's metrization theorem [3], i.e., for each n, G'_n is an open covering of X, $G'_{n+1} \subseteq G'_n$, and for every $p \in X$, $\{St^2(G'_n, p) : n < \omega\}$ forms a local base for the topology at p. For each $p \in H$, denote by $R_0(p)$ an element of G'_0 containing p whose closure is a subset of $T(p)$ and, having defined $R_{n-1}(p)$, denote by $R_n(p)$ an element of G'_n containing p whose closure is a subset of $R_{n-1}(p)$. For each $n < \omega$, let G_n be the collection of all elements g of G'_n such that if $p \in H$ and $i \leq n$ then g does not intersect both $R_i(p)$ and $X\backslash R_{i-1}(p)$.

Let ρ denote a metric on Y. For $y \in Y$ and $\delta > 0$ let $B(y, \delta) = \{z \in Y : \rho(z, y) < \delta\}$. If $q \in Y$ and $k < \omega$, define

$$E_k(q) = B(q, 2^{-k}) \cup \bigcup \{R_k(T_j(s)) : k \leq j < \omega \text{ and } s \in B(q, 2^{-k}) \cap K\}.$$

For $i < \omega$ define $M_i = \{E_j(q) : q \in Y \text{ and } j \geq i\}$. Let $L_i = M_i \cup G_i$, and let $Z = X \cup Y$. Then L_0 is a basis for a T_1-topology Ω on Z and $\{L_n : n < \omega\}$ satisfies the conditions of Moore's theorem, so that (Z, Ω) is metrizable. Clearly, the inclusion maps $\Phi_X : X \to Z$ and $\Phi_Y : Y \to Z$ are homeomorphisms, X is a dense open set in Z, and $Z\backslash X = Y$.

Now, let Z' be a completely metrizable space containing Z. Since X and $Z\backslash X$ are completely metrizable, they are G_δ-sets in Z'. The union of two G_δ-sets is a G_δ-set, so Z is a G_δ-set in the complete space Z' and is itself complete.

Corollary 1. *If X and Y are completely metrizable, X is not compact, and Y is separable, then Y is a completion remainder of X.*

Remark. It is clear that if X and Y are metrizable and there is a dense embedding h of X into a completely metrizable space Z such that $Z\backslash h(X) \simeq Y$ and such that $h(X)$ is open in Z, then X and Y must be completely metrizable. It is also clear that the metrizable space X is an absolute F_σ if and only if every completion remainder of X is complete. Thus we have

Corollary 2. *The completion remainders of* $\mathbf{R}$, *or, indeed, of any separable, locally compact, noncompact, metrizable space, are the nonempty Polish spaces.*

Corollary 3. *Suppose X and Y are completely metrizable and X is not compact. Let $e(X)$ be achieved, i.e., let X have a closed, discrete set of cardinality $e(X)$. Then the following are equivalent.*

(A) $d(Y) \leq e(X)$.
(B) *Y is a completion remainder of X.*
(C) *Y is a closed completion remainder of X.*

Theorem 2. *Suppose X and Y are completely metrizable and X is not compact. Suppose $e(X)$ is not achieved. Then Y is a completion remainder of X if and only if $d(Y) \leq e(X)$ and $ld_y(Y) < e(X)$ for every $y \in Y$.*

Proof. Let $\{\alpha_n : n < \omega\}$ be an increasing sequence of cardinals whose supremum is $e(X)$. As before, we can assume that $X \cap Y = \varnothing$. Suppose first that Z is completely metrizable, $Z = X \cup Y$, and X is dense in Z. Suppose there is a point $y \in Y$ such that $ld_y(Y) = e(X)$. For each open set U in Z containing y, $d(U \cap X) = e(X)$. Take a sequence $\{U_n : n < \omega\}$ of open sets in Z, $\overline{U_{n+1}} \subset U_n$, $y \in U_n$ for every $n < \omega$, and $\operatorname{diam}(U_n) < 2^{-n}$. For each $n < \omega$, there is a closed discrete set D_n in $X \cap \overline{U_n}$, $|D_n| = \alpha_n$. Then $D = \bigcup_{n<\omega} D_n$ is closed and discrete in X and $|D| = e(X)$, a contradiction. This completes the necessity proof.

Next, suppose $d(Y) \leq d(X)$ and $ld_y(Y) < e(X)$ for all $y \in Y$. As before, we assume that $X \cap Y = \varnothing$. We first consider the special case in which $Y = \bigcup_{n<\omega} Y_n$ is a countable discrete union of closed subsets, each of density less than $e(X)$. We show in this case that Y is a completion remainder of X in such a way that in the completion $Z = X \cup Y$, the sequence $\{Y_n : n < \omega\}$ converges to a point p of X.

Let p be a point of X such that $ld_p(X) = d(X)$. Let U_0 be an open set in X containing p with $\operatorname{diam} U < 2^{-0}$. There is a closed discrete set D_0 in X, $D_0 \subseteq U_0$, $p \notin D_0$, $|D_0| = \alpha_0$. There exists a discrete collection $\mathscr{G}_0$ of open sets screening $D_0 \cup \{p\}$ such that $\bigcup\{\overline{G} : G \in \mathscr{G}_0\} \subset U_0$. Let $G_{0,p}$ be the element of $\mathscr{G}_0$ containing p. Having chosen U_{n-1}, D_{n-1}, $\mathscr{G}_{n-1}$, and $G_{n-1,p}$, take U_n to be an open set in X, $p \in U_n$, $\operatorname{diam} U_n < 2^{-n}$, $\overline{U}_n \subset G_{n-1,p}$; take D_n to be a closed discrete set in X, $D_n \subseteq U_n$, $p \notin D_n$, $|D_n| = \alpha_n$; take $\mathscr{G}_n$ to be a discrete collection of open sets screening $D_n \cup \{p\}$ such that $\bigcup\{\overline{G} : G \in \mathscr{G}_n\} \subseteq U_n$; and let $G_{n,p}$ denote the element of $\mathscr{G}_n$ containing p.

As in the sufficiency proof of Theorem 1, there is a topology T_n on $(U_n \backslash \overline{U}_{n+1}) \cup Y_n$ such that Y_n is closed and nowhere dense in $(U_n \backslash \overline{U}_{n+1}) \cup Y_n$, and where $\mathscr{G}_n$ plays the role of the discrete collection $\mathscr{H}$. This latter condition implies that there is a base for T_n that is σ-discrete in X. For each $n < \omega$, let $U_n^* = U_n \cup \bigcup\{Y_n : m \geq n\}$. Then $T = (\bigcup_{n<\omega} T_n) \cup \{U_n^* : n < \omega\} \cup \{U : U$ open in X, $p \notin U\}$ is a basis for a completely metrizable topology on $X \cup Y$, $X \backslash \{p\}$ is dense and open in $X \cup Y$, and $\{Y_n : n < \omega\}$ converges in $X \cup Y$ to $\{p\}$.

We now proceed to the general case. There is a minimal, locally finite open cover $\mathscr{U}$ of Y such that if $U \in \mathscr{U}$ then $d(U) < e(X)$. Now, $|\mathscr{U}| \leq d(Y) \leq e(X)$, so $\mathscr{U}$ is a countable union, $\mathscr{U} = \bigcup_{n<\omega} \mathscr{U}_n$, where $|\mathscr{U}_n| \leq \alpha_n$. Let $\mathscr{U}_{n,m} = \{U \in \mathscr{U}_n : d(U) \leq \alpha_m\}$, and let $U_{n,m} = \bigcup \mathscr{U}_{n,m}$. Note that $d(U_{n,m}) \leq \alpha_n \cdot \alpha_m < e(X)$. By Engelking [1, Lemma 5.2.4], the countable open cover this cover can be shrunk, $\{\mathrm{Cl}(V_n) : n < \omega\}$ can be taken to be a star-finite cover of Y. For $n < \omega$, let $Y_n = \overline{V}_n$, let $\{Y_n' : n < \omega\}$ be a sequence of disjoint spaces, $Y_n' \simeq Y_n$, and let Y^* be the free union of the Y_n''s. Then there is a completely metrizable topology on $X \cup Y^*$ as in the special case, with $\{Y_n' : n < \omega\}$ converging to a point $p \in X$. Let $f : X \cup Y^* \to X \cup Y$ be the obvious quotient map. Note that $f^{-1}(q)$ is finite for all $q \in X \cup Y$.

We claim that f is a closed and therefore perfect mapping. For, let $H \subseteq X \cup Y^*$ be closed. If $p \notin H$ then $H \cap Y_n' = \varnothing$ for all sufficiently large n. Since $f^{-1}(f(H)) = H \cup \bigcup_{n,m<\omega} f^{-1}(f(H \cap Y_n') \cap Y_m)$, it follows that $f(H)$ is closed in this case. But it is also clearly closed if $p \in H$. Thus $X \cup Y$ is completely metrizable, since it is a perfect image of a completely metrizable space. Clearly, X is dense in $X \cup Y$.

3. Completion remainders of $\mathbf{Q}$ and of $\mathbf{P}$

Theorem 3. *The completion remainders of* $\mathbf{Q}$ *are the nowhere locally compact Polish spaces.*

Proof. Assume Y is a nowhere locally compact Polish space, regarded as a subset of the Hilbert cube. Let $K = \overline{Y}$. Then Y is a dense G_δ-set in the compact metric space K, and, since Y is nowhere locally compact, $K \backslash Y$ is dense in K. Let $G = \{G_n : n < \omega\}$ be a countable basis for K. For $n < \omega$, let U_n be open in K, with $Y = \bigcap_{n<\omega} U_n$, $U_n \supseteq U_{n+1}$. Choose $a_n \in (U_n \backslash Y) \cap G_n$. Let $A = \{a_n : n < \omega\}$. Then A, being a countable metric space with no isolated points, is homeomorphic to $\mathbf{Q}$. Let $Z = A \cup Y$. Then A is dense in Z. It remains to be shown that Z is completely metrizable. We show that Z is a G_δ-set in K. Let $V_n = U_n \cup \{a_i : i < n\}$. Each V_n, as the union of two G_δ-sets, is a G_δ-set, so V_n is one and $Z = \bigcap_{n<\omega} V_n$ is thus a G_δ-set.

Next, assume that Y is a remainder of $\mathbf{Q}$. Let $Z = A \cup Y$, where Z is completely metrizable, $A \simeq \mathbf{Q}$, $A \cap Y = \varnothing$, A is dense in Z. It is immediate that Y is a Polish space. Suppose Y is locally compact at some point $p \in Y$. Let U_Y be an open set in Y containing p, $J = \mathrm{Cl}_Y(U_Y)$ is compact. Then J is closed in Z. There is an open set U in Z such that $U \cap Y = U_Y$. Then $U \backslash J$ is open in Z and therefore topologically complete. But $U \backslash J$ is a subset of A with no isolated point, so $U \backslash J \simeq \mathbf{Q}$, a contradiction.

For the sake of completeness, we include the following. The proofs are immediate.

Theorem 4. *The topological completions of* $\mathbf{Q}$ *and those of* $\mathbf{P}$ *are the Polish spaces with no isolated points.*

Since $\mathbf{Q}$ is a completion remainder of $\mathbf{P}$, the irrationals, one might wonder whether every σ-compact metric space is a completion remainder of $\mathbf{P}$. That this is not the case is shown in

Theorem 5. *If the metrizable space* S *contains a nondegenerate continuum then* $\mathbf{Q} \times S$ *is not a completion remainder of* $\mathbf{P}$.

Proof. We may assume that S is separable. Let I be a nondegenerate continuum in S. Suppose $\Theta : \mathbf{Q} \times S \to Z$ is an embedding, where Z is a Polish space. We will prove the theorem by showing that $Z \backslash \Theta(\mathbf{Q} \times S)$ contains a nondegenerate connected set and hence is not homeomorphic to $\mathbf{P}$. Z is a dense G_δ-set in some compact metric space K. Let $K\backslash Z = \bigcup_{n<\omega} K_n$, where each K_n is compact. For each $t \in \mathbf{R}$, let

$$L_t = \bigcap_{n<\omega} \mathrm{Cl}_K(\Theta((t-2^{-n}, t+2^{-n}) \cap \mathbf{Q}) \times I).$$

Note that if $t \in \mathbf{Q}$ then $L_t = \Theta(\{t\} \times I)$, and if $t \notin \mathbf{Q}$ then $L_t \cap \Theta(\mathbf{Q} \times S) = \varnothing$.

Let $W_n = \{t \in \mathbf{R} : L_t \cap K_n \neq \varnothing\}$. Suppose some W_k is dense in some open interval (a, b) in $\mathbf{R}$. Let $q \in (a, b) \cap \mathbf{Q}$, and let $t_n \in W_k$ with $t_n \to q$. Let $x_n \in L_{t_n} \cap K_k$. There is a limit point x of $\{x_n : n \in \omega\}$ in K_k. Clearly $x \in L_q = \Theta(\{q\} \times I) \subset Z$, a contradiction. Thus each W_k is nowhere dense.

Now pick distance points s and p of I, pick $q \in \mathbf{Q}$, and let

$$\delta = \rho_K(\Theta(q, p), \Theta(q, s)),$$

where ρ_K is a metric on K. Note that $M = \{r \in \mathbf{Q} : \rho_K(\Theta(r, p), \Theta(r, s)) > \delta/2\}$ is a nonempty open subset of $\mathbf{Q}$. Let $t \in \overline{M}\backslash(\mathbf{Q} \cup \bigcup_{n<\omega} W_n)$. Then $L_t \cap K_n = \varnothing$ for all $n < \omega$, i.e., $L_t \subset Z$. Since $t \notin \mathbf{Q}$, $L_t \subset Z\backslash\Theta(\mathbf{Q} \times S)$. Choose $q_n \in M$, $q_n \to t$. By passing to a subsequence if necessary, we may assume that $\{\Theta(\{q_n\} \times I) : n \in \omega\}$ converges in the Vietoris topology on 2^K to some set J, which is nondegenerate, connected, and contained in L_t. Thus $Z\backslash\Theta(\mathbf{Q} \times S)$ contains a nondegenerate connected set and so is not homeomorphic to $\mathbf{P}$.

Corollary 4. $\mathbf{Q} \times \mathbf{R}$ *is not a completion remainder of* $\mathbf{P}$.

Remark. The following generalization has essentially the same proof as that of Theorem 5.

Theorem 6. *If X contains a closed subset Y such that there exists an open and closed mapping $f : Y \to \mathbf{Q}$ such that each $f^{-1}(q)$ is a nondegenerate continuum, then X is not a completion remainder of* $\mathbf{P}$.

Theorem 7. *Every σ-compact,* 0*-dimensional metrizable space is a completion remainder of* $\mathbf{P}$.

Proof. Suppose Y is as in the hypothesis. There is an embedding $\phi: Y \to \mathbf{P}$. $\mathbf{P}\backslash\phi(Y)$ is a G_δ-set in $\mathbf{P}$; it is separable, 0-dimensional, metrizable, and nowhere locally compact, so it is homeomorphic to $\mathbf{P}$.

4. Completion remainders of $\mathbf{Q}(\kappa)$ and $\mathbf{P}(\kappa)$

Throughout this section κ denotes an infinite cardinal. Let $\mathbf{Q}(\kappa)$ denote a σ-discrete metric space in which every open set has cardinality κ. Medvedev [2] has shown that all such spaces are homeomorphic. Let $\mathbf{P}(\kappa)$ denote a complete metric space with covering dimension 0 that has density κ and local density κ at each point and that is nowhere locally κ-compact. A straightforward argument shows that all such spaces are homeomorphic; in particular, $\mathbf{P}(\kappa)$ is homeomorphic to the Baire space $B(\kappa)$, the countable Cartesian product of discrete spaces of cardinality κ. It follows that $\mathbf{P}(\kappa)$ is a completion remainder of $\mathbf{Q}(\kappa)$. The 0-dimensionality, however, is not necessary. We have

Theorem 8. *The completion remainders of $\mathbf{Q}(\kappa)$ are the completely metrizable spaces that have density κ and local density κ at every point but that are nowhere locally κ-compact.*

Before proving Theorem 8 we present a lemma that extends the old result of Niemytzki and Tchyonoff [4] that a metrizable space is compact if and only if every compatible metric on the space is complete.

Lemma 2. *Let X be a metrizable space. Then the following are equivalent.*

(A) *X is nowhere locally compact.*

(B) *X can be embedded in a metrizable space Z in such a way that both X and $Z\backslash X$ are dense in Z.*

(C) *X admits a compatible metric that is nowhere locally complete.*

Proof. (B) $\Rightarrow$ (C) Let X and Z be as in (B); Z can be densely embedded in a complete metric space $\langle W, \rho\rangle$. Then X is dense in W, and if ρ_X denotes the restriction of ρ to $X \times X$ then X is nowhere locally complete according to ρ_X.

(C) $\Rightarrow$ (A) The proof follows immediately from the Niemytzki-Tychonoff Theorem.

(A) $\Rightarrow$ (B) Suppose X is nowhere locally compact. Let ρ be a metric on X. We make repeated use of the following observation.

($*$) For every nonempty open set U in X, there is a sequence $\{U_n : n < \omega\}$ of nonempty open sets in X such that $\overline{U_0} \subset U$, $\overline{U_{n+1}} \subset U_n$ for all $n < \omega$, and $\bigcap_{n<\omega} U_n = \varnothing$.

There exists a locally finite open cover G_0 of X such that if $g \in G_0$ then ρ-diam $g < 2^{-0}$ and g contains a point not in $\bar{h}$ for any $h \in G_0\backslash\{g\}$.

For each $g \in G_0$ let U_g be a nonempty open set such that $\overline{U}_g \subset g$ and $\overline{U}_g \cap \bar{h} = \varnothing$ for every $h \in G_0\backslash\{g\}$. Let $\{U_n(g) : n < \omega]$ be a sequence as in ($*$), with $\overline{U_0(g)} \subset U_g$.

Take $G_0' = G_0 \cup \{U_n(g) : g \in G_0,\ n < \omega\}$. If $x \in X$ there is an open set $v_0(x)$ containing x that intersects only finitely many elements of G_0'. Let $V_0 = \{v_0(x) : x \in X\}$.

There exists, for each n, $0 < n < \omega$, collections G_n, G_n', V_n, $\{U_g : g \in G_n\}$, $\{U_m(g) : g \in G,\ m < \omega\}$ such that

(1) G_n is a locally finite open cover of X and ρ-diam $g < 2^{-n}$ for all $g \in G_n$.

(2) G_n refines both G_{n-1}' and V_{n-1}.

(3) If $g \in G_n$ then U_g is a nonempty open set such that

(i) $\overline{U}_g \subset g$ and $\overline{U}_g \cap \bar{h} = \varnothing$ for every $h \in G_n\backslash\{g\}$;

(ii) if $h \in G_0 \cup \cdots \cup G_{n-1}$ and $U_g \cap U_m(h) \neq 0$, $m < \omega$, then $\overline{U}_g \subset U_m(h)$; and

(iii) if $h \in G_0 \cup \cdots \cup G_{n-1}$ then there is an $m < \omega$ such that $\overline{U}_g \cap \overline{U_m(h)} = \varnothing$.

(4) If $g \in G_n$ then $\{U_m(g) : m < \omega\}$ is as in ($*$) with $\overline{U_0(g)} \subset U_g$.

(5) $G_n' = G_{n-1}' \cup G_n \cup \{U_m(g) : g \in G_n,\ m < \omega\}$,

(6) V_n is an open cover of X no element of which intersects infinitely many elements of G_n'.

Let $A = \bigcup_{n<\omega} A_n$, where $A_n \cap A_m = \varnothing$ for $n \neq m$, $A \cap X = \varnothing$, and, for each n, $|A_n| = |G_n|$. Let $\phi_n : A_n \to G_n$ be a bijection. Let $Z = A \cup X$. For V open in X let $A_V = \{a \in A :$ for some $m, n < \omega$, $a \in A_n$, and $\mathrm{Cl}_X(U_m(\phi_n(a))) \subset V\}$, and let $E(V) = V \cup A_V$. We observe that $\{E(V) : V$ open in $X\}$ is a cover of Z and that if V and W are open in X then

$E(V \cap W) = E(V) \cap E(W)$. Therefore, $\{E(V) : V \text{ open in } X\}$ is a basis for a topology Ω on Z. We also observe that $E(V) \subset E(W)$ whenever $V \subset W$ and that $\mathrm{Cl}_Z(E(V)) = \mathrm{Cl}_Z(V)$ for all open V in X. It is easily seen that Ω is a Hausdorff topology on Z. We list three more observations that are useful in showing Ω is regular.

(1) $\mathscr{U} = \{U_m(a) : m < \omega,\ a \in A\}$ is non-Archimedean in the sense that if two members of $\mathscr{U}$ intersect then one is a subset of the other.

(2) If $p \in A_n$ and $q \in A \cap \mathrm{Cl}_Z(U_m(\phi_n(p)))$ then $q \in E(U_m(\phi_n(p)))$.

(3) If $p \in A_n$ then $\mathrm{Cl}_Z(E(U_{m+1}(\phi_n(p)))) \subset E(U_m(\phi(p)))$.

Next, assume $p \in A$ and $E(U)$ is a basic open set containing p. There is an n such that $p \in A_n$. There is an m such that $\mathrm{Cl}_X(U_m(\phi_n(p))) \subset U$. Let $V = E(U_{m+1}(\phi_n(p)))$. Then by (3) above, $\mathrm{Cl}_Z(V) \subset E(U_m(\phi_n(p))) \subset E(U)$. Therefore, Ω is regular at points of A.

Next, assume $p \in X$ and $E(U)$ is a basic open set containing p. There is an n such that if $p \in g \in G_n$, $h \in G_n$, and $g \cap h \neq \varnothing$, then $h \subset U$. Choose an element g of G_n containing p. For each $i \leq n$ there is an $m_i < \omega$ such that $p \notin \mathrm{Cl}_X(U_{m_i}(\phi_i(a)))$ for any $a \in A_i$. There is an open set V in X, with $p \in V \subset g$, and $\mathrm{Cl}_X(V) \cap \mathrm{Cl}_X(U_{m_i}(\phi_i(a))) = \varnothing$ for all $a \in A_0 \cup \cdots \cup A_n$. So, if $a \in A_0 \cup \cdots \cup A_n$, then $a \notin \mathrm{Cl}_Z(V)$. Suppose $a \in A_k$, $k > n$, and $a \in \mathrm{Cl}_Z(E(V)) = \mathrm{Cl}_Z(V)$. Then $V \cap E(U_0(a))) \neq \varnothing$ and $U_0(\phi_k(a))$ is a subset of some h in G_n, and $h \cap g \neq \varnothing$; so $h \subset U$, which implies $a \in E(U)$. Therefore, Ω is regular at points of X.

Next, we exhibit a σ-locally finite basis for Ω. Let $\Sigma_0 = \{E(g) : g \in G_0\}$. Then Σ_0 is locally finite. For $1 \leq n < \omega$ and $k < \omega$, let $\Sigma(n, k) = \{E(g) : g \in G_n,\ U_k(h) \cap g = \varnothing \text{ for all } h \in G_0 \cup \cdots \cup G_{n-1}\}$. Then $\Sigma(n, k)$ is locally finite. Moreover, if $\Sigma_n = \{E(g) : g \in G_n\}$, then $\Sigma_n = \bigcup_{k<\omega} \Sigma(n, k)$.

Similarly, it follows that for all $m, n < \omega$, $\Delta_{m,n} = \{E(U_m(g)) : g \in G_n\}$ is σ-locally finite.

Then $(\bigcup_{n<\omega} \Sigma_n) \cup (\bigcup_{m,n<\omega} \Delta_{m,n})$ is a σ-locally finite basis for Ω, so that (Z, Ω) is metrizable by the Nagata-Smirnov theorem.

Clearly, A is dense in Z and so is X. This completes the proof of Lemma 2.

We now return to the proof of Theorem 8. Note that Theorem 3 is Theorem 8 in the special case $\kappa = \omega$. From now on we assume $\kappa > \omega$.

Assume Y is a completely metrizable space with density κ and local density κ at each point. It follows directly from Lemma 1 that Y is nowhere locally κ-compact and therefore nowhere locally compact. We apply Lemma 2 to get a metrizable space Z such that both Y and $Z \backslash Y$ are dense in Z. We may assume that Z is completely metrizable, since it can be densely embedded in a completely metrizable space Z', and that both Y and $Z' \backslash Y$ are dense in Z'.

Let $G = \bigcup_{n<\omega} G_n$ be a σ-discrete basis for Z, where $|G_n| = \kappa$ and G_n is discrete, $n < \omega$.

Since Y is completely metrizable, it is a G_δ-set in Z; let $\{V_n : n < \omega\}$ be a sequence of open sets in Z, with $V_n \supset V_{n+1}$, $\bigcap_{n<\omega} V_n = Y$.

For each $n < \omega$, let A_n be an Axiom of Choice set for $\{(g \cap V_n) \backslash Y : g \in G_n\}$. Then A_n is closed and discrete and $A = \bigcup_{n<\omega} A_n$ is σ-discrete and has density κ and local density κ at every point. It follows that $A \simeq \mathbf{Q}(\kappa)$. Moreover, A is dense in Z. For $n = 0$, let $W_0 = V_0$, and for $n > 0$, let

$W_n = A_0 \cup \cdots \cup A_{n-1} \cup V_n$. Since each A_i is closed and V_n is open, W_n is a G_δ-set in Z, so $A \cup Y = \bigcap_{n<\omega} W_n$ is a G_δ-set in Z and therefore completely metrizable.

Next assume that Y is a completion remainder of $\mathbf{Q}(\kappa)$. Then there exist A and Z, $A \simeq \mathbf{Q}(\kappa)$, Z completely metrizable, $Z = A \cup Y$, and $A \cap Y = \varnothing$. Since A is an absolute F_σ, Y is a G_δ-set in Z and thus completely metrizable. Since A is dense in Z, we have $d(Y) \le d(Z) \le d(A) = \kappa$. Since Y is dense in Z, we have $\kappa = d(A) \le d(Z) \le d(Y)$. Therefore, $d(Y) = \kappa$. Similarly, $ld_p(Y) = \kappa$ for each $p \in Y$. It follows from Lemma 1 that Y is nowhere locally κ-compact.

Theorem 9. *If Y is metrizable,* $\dim Y = 0$, *and Y is the union of countably many sets, each the union of a discrete collection of compact sets, then Y is a completion remainder of* $\mathbf{P}(\kappa)$.

Proof. The proof is very similar to that of Theorem 6. Firstly, we know that there is an embedding $\Phi\colon Y \to \mathbf{P}(\kappa)$. Secondly, $\mathbf{P}(\kappa)\backslash\Phi(Y)$ is a G_δ-set in $\mathbf{P}(\kappa)$; it has covering dimension 0 and density κ and local density κ at every point and is nowhere locally κ-compact, so it is homeomorphic to $\mathbf{P}(\kappa)$.

5. Open questions

Question 1. What are the completion remainders of $\mathbf{P}(\kappa)$? We do not have a characterization even in case $\kappa = \omega$.

Question 2. In the class of Moore spaces, what are the completion remainders of $\mathbf{Q}$ or of $\mathbf{Q}(\kappa)$?

Acknowledgments

We are indebted to I. Reclaw, S. Baldwin, and the referee for helpful suggestions.

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(B. Fitzpatrick, Jr., and G. F. Gruenhage) Division of Mathematics, FAT, Auburn University, Auburn, Alabama 36849-5310

E-mail address: GGRUEN@DUCVAX.auburn.edu

(J. W. Ott) Triangle Systems, Inc., Chapel Hill, North Carolina 27709