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REDUCIBLE YANG-MILLS CONNECTIONS ON KÄHLER SURFACES AND MOMENT MAPS

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ABSTRACT. We determine the second-order approximation of the anti-self-duality equation around a reducible Yang-Mills connection on a compact 1-connected Kähler surface.

The goal of this note is to explain how one can derive an approximation of the anti-self-duality (ASD) equation around a $U(1)$-reduction on a compact simply connected Kähler surface. The idea is to study two different moment map models associated to such a reduction. This observation is due to Donaldson as he explained to the author his understanding of a result in [M]. To begin with, we recall first the following general facts about a reducible ASD connection A on a smooth compact simply connected oriented 4-manifold X. We work with a fixed metric m_0 on X throughout this discussion.

Suppose A is a reducible ASD connection on an $SU(2)$-bundle $P \to X$ preserving a splitting $L \oplus L^{-1}$ for some line bundle $L \to X$, where $c_2(P) = -L \cdot L = k > 0$. It is a well-known fact that a neighbourhood of $[A] \in M_k(m_0)$, the moduli space of equivalence classes of m_0-anti-self-dual connections on P, can be modelled as an S^1-quotient $\phi^{-1}(0)/S^1$ for some finite-dimensional equivariant map

$$\phi\colon H_A^1 \to H_A^2 \tag{1.1}$$

defined on a small neighbourhood of $O \in H_A^1$. (See for instance [L].) Here we write H_A^i, $i = 0, 1, 2$, for the cohomology groups of the Atiyah-Hitchin-Singer deformation complex

$$0 \to \Omega^0(\operatorname{ad} P) \xrightarrow{d_A} \Omega^1(\operatorname{ad} P) \xrightarrow{d_A^+} \Omega^2_+(\operatorname{ad} P) \to 0$$

associated to the ASD connection A. More precisely in (1.1), we find a smooth map

$$v \in H_A^1 \to \tilde{v} \in \ker d_A^* \subset \Omega^1(\operatorname{ad} P)$$

modelled on suitable Hilbert spaces so that for $|v| \ll 1$ the map $\tilde{v}$ solves

$$\phi(v) = F_+(A + \tilde{v}) \in H_A^2$$

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with

$$|\tilde{v} - v| \le \text{const}\,|v|^2 \tag{1.2}$$

(cf. [D]). Here $F_+(A+\tilde{v})$ denotes the self-dual curvature associated to $A+\tilde{v}$. It is clear from (1.1) and (1.2) the map ϕ satisfies

$$\phi(0) = 0, \qquad d\phi(0) = 0,$$

and so it is of interest to identify the second-order approximation of the map ϕ about $O \in H^1_A$. This can be achieved on a simply connected Kähler surface provided certain assumptions are made. (See (1.6) for details.)

In order to explain this, we pass the above discussion to a compact simply connected Kähler surface Y. So assume now the metric m_0 on Y is Kähler and $L \to Y$ denotes a holomorphic line bundle satisfying

$$L \cdot L = -k \quad \text{and} \quad \omega_0 \cdot L = 0,$$

where ω_0 is the Kähler form on Y associated to m_0. Given that Y is a Kähler surface, one recalls there is defined a moment map $\mu\colon \mathscr{A} \to \Omega^4(\operatorname{ad} P)$ for the gauge group $\mathscr{G}$ action on $\mathscr{A}$, the space of connections on P (cf. [AB]). A point of introducing this map is that the zero set $\mu^{-1}(0)$ in $\mathscr{A}$ contains precisely m_0-ASD connections on P since $\mu(A) = F_+(A) \wedge \omega_0$ by a direct calculation. This interesting relation between the moment map μ and the ASD equation leads us to wonder if there is a role for a moment map in the finite-dimensional model (1.1) for the ASD equation. The point is that if A is a reducible connection on P, then it is well known that H^1_A is a direct sum of complex spaces $\mathbb{C}^p$ and $\mathbb{C}^q$ for some p, $q \ge 0$ and that the isotropy group $\Gamma_A \simeq S^1 \subset \mathscr{G}$ of A acts on these complex spaces with weights 2 on $\mathbb{C}^p$ and -2 on $\mathbb{C}^q$ (cf. [L]). One may then consider the moment map

$$\mu_0\colon H^1_A \simeq \mathbb{C}^p \oplus \mathbb{C}^q \to iR,$$
$$(z_1, \dots, z_p; w_1, \dots, w_q) \mapsto \frac{i}{2}\left\{\sum_{\alpha=1}^{p} |z_\alpha|^2 - \sum_{\beta=1}^{q} |w_\beta|^2\right\}$$

associated to this group action and ponder if there is a relation between μ_0 and ϕ, the map in (1.1) modelling the ASD equation near A. Our main objective here is to exploit this observation and show under appropriate assumptions on Y and L that the map μ_0, if suitably put, is precisely the second-order approximation of ϕ.

To be more precise, we assume in the following discussion that ϕ takes a particular simple form

$$\phi\colon \{v \in H^1_A \,|\, |v| \ll 1\} \to \mathbb{R} \cdot \begin{pmatrix} i\omega_0 & 0 \\ 0 & -i\omega_0 \end{pmatrix}, \tag{1.3}$$

but this assumption imposes conditions on Y and L. To see this, we note that working over complex manifolds one can associate to the connection A a twisted Cauchy-Riemann operator

$$\bar{\partial}_A\colon \Omega^{0,0}(\operatorname{ad} P) \to \Omega^{0,1}(\operatorname{ad} P)$$

and define thereby Dolbeault cohomology groups $H^{0,i}_{\bar{\partial}_A}$, $i = 0, 1, 2$, in a natural way. As Y is Kähler, there are natural isomorphisms

$$H_A^1 \simeq H_{\bar\partial_A}^{0,1} \tag{1.4}$$

and

$$H_A^2 \simeq H_{\bar\partial_A}^{0,2} \oplus H_A^0 \tag{1.5}$$

relating cohomology groups of these two kinds (cf. [K, p. 248]). Here in our discussion, one interprets

$$H_A^0 \simeq \mathbb{R} \cdot \begin{pmatrix} i\omega_0 & 0 \\ 0 & -i\omega_0 \end{pmatrix}$$

in (1.5). Thus the map ϕ in (1.3) takes the stated form if the part $H_{\bar\partial_A}^{0,2}$ in (1.5) vanishes, and this is the case if

$$\text{(i) } H_{\bar\partial}^{0,2}(Y) = 0, \qquad \text{(ii) } H_{\bar\partial_A}^{0,2}(L^{\pm 2}) = 0, \tag{1.6}$$

conditions on Y and L we shall assume from now on. Note that condition (i) is equivalent to $b_2^+(Y) = 1$.

Using such a simple description of ϕ, we can define a dual map

$$\begin{aligned} &\hat\phi\colon H_A^1 \to \mathbb{R}, \\ &v \mapsto -\int_Y \operatorname{Tr}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \phi(v)\right) \wedge \omega_0 \end{aligned}$$

on $\{|v| \ll 1\}$ having the property that

$$\phi(v) = \frac{\hat\phi(v)}{4\operatorname{vol}(Y)} \cdot \begin{pmatrix} i\omega_0 & 0 \\ 0 & -i\omega_0 \end{pmatrix} \in \mathbb{R} \cdot \begin{pmatrix} i\omega_0 & 0 \\ 0 & -i\omega_0 \end{pmatrix}.$$

Clearly then $\hat\phi^{-1}(0)/S^1$ provides an alternative model for a small neighbourhood of $[A] \in M_k(m_0)$. Similar to the map ϕ, one finds $\hat\phi$ satisfies $\hat\phi(0) = d\hat\phi(0) = 0$, and so we study the second-order approximation of $\hat\phi$ about $O \in H_A^1$. Let

$$\hat\phi_0(v) = -\int_Y \operatorname{Tr}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, F_+(A+v)\right) \wedge \omega_0 \tag{1.7}$$

where $v \in H_A^1$ with $|v| \ll 1$.

(1.8) **Lemma.** *On small neighbourhoods $\{|v| \ll 1\}$ of $O \in H_A^1$, the function $\hat\phi$ is approximated by $\hat\phi_0$ in the sense that*

$$\hat\phi(v) = \hat\phi_0(v) + O(|v|^3).$$

Proof. Assuming $p = \tilde v - v$, one finds

$$F_+(A + \tilde v) = F_+(A+v) + d_A^+ p + (v \wedge p + p \wedge v + p \wedge p)_+,$$

and hence that

$$\begin{aligned} -\int_Y &\operatorname{Tr}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, F_+(A+\tilde v)\right) \wedge \omega_0 \\ &= -\int_Y \operatorname{Tr}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, F_+(A+v)\right) \wedge \omega_0 \\ &\quad -\int_Y \operatorname{Tr}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, d_A^+ p\right) \wedge \omega_0 + O(|v|^3) \end{aligned}$$

as $|p| \le \text{const}\,|v|^2$ by (1.2). Now (1.8) follows should one notice

$$-\int_Y \operatorname{Tr}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, d_A^+ p\right) \wedge \omega_0 = \int_Y \operatorname{Tr}\left(d_A \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \wedge p\right) \wedge \omega_0 = 0$$

since $d_A\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) = 0$ relative to the splitting $L \oplus L^{-1}$. This completes the proof.

Now we identify $\hat{\phi}_0$. For this purpose, observe first relative to the splitting $L \oplus L^{-1}$ the Dolbeault cohomology group $H^{0,1}_{\bar{\partial}_A}$ is naturally a direct sum of Hermitian vector spaces

$$H^{0,1}_{\bar{\partial}_A} \simeq H^{0,1}_{\bar{\partial}_A}(L^2) \oplus H^{0,1}_{\bar{\partial}_A}(L^{-2}).$$

By taking two sets of unitary bases, say,

$$\{\varphi_\alpha | 1 \le \alpha \le h^1(L^2)\} \quad \text{and} \quad \{\psi_\beta | 1 \le \beta \le h^1(L^{-2})\},$$

for $H^{0,1}_{\bar{\partial}_A}(L^2)$ and $H^{0,1}_{\bar{\partial}_A}(L^{-2})$, respectively, one finds

$$H^{0,1}_{\bar{\partial}_A} \simeq \left\{ \sum_{\alpha=1}^{h^1(L^2)} Z_\alpha \begin{pmatrix} 0 & \varphi_\alpha \\ 0 & 0 \end{pmatrix} + \sum_{\beta=1}^{h^1(L^{-2})} W_\beta \begin{pmatrix} 0 & 0 \\ \psi_\beta & 0 \end{pmatrix} \middle| Z_\alpha, W_\beta \in \mathbb{C} \right\}.$$

Furthermore, via the isomorphism $H^1_A \simeq H^{0,1}_{\bar{\partial}_A}$, we obtain in turn a (real) basis for H^1_A:

$$a_\alpha = \begin{pmatrix} 0 & \varphi_\alpha \\ -\bar{\varphi}_\alpha & 0 \end{pmatrix}, \quad Ia_\alpha = \begin{pmatrix} 0 & i\varphi_\alpha \\ -\overline{i\varphi_\alpha} & 0 \end{pmatrix}, \quad \alpha = 1, \dots, h^1(L^2);$$
$$b_\beta = \begin{pmatrix} 0 & -\bar{\psi}_\alpha \\ \psi_\beta & 0 \end{pmatrix}, \quad Ib_\beta = \begin{pmatrix} 0 & -\overline{i\psi_\beta} \\ i\psi_\beta & 0 \end{pmatrix}, \quad \beta = 1, \dots, h^1(L^{-2}).$$

In these notations, it is not difficult to check every vector $v \in H^1_A$ can be uniquely written as a combination

$$\begin{aligned} v &= \begin{pmatrix} 0 & \sum Z_v^\alpha \varphi_\alpha \\ -\sum \overline{Z_v^\alpha}\,\overline{\varphi_\alpha} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sum \overline{W_v^\beta}\,\overline{\psi_\beta} \\ \sum W_v^\beta \psi_\beta & 0 \end{pmatrix} \\ &= \sum (\operatorname{Re} Z_v^\alpha a_\alpha + \operatorname{Im} Z_v^\alpha I a_\alpha) + \sum (\operatorname{Re} W_v^\beta b_\beta + \operatorname{Im} W_v^\beta I b_\beta) \end{aligned}$$

for some complex scalars Z_v^α, W_v^β. Now we can describe the approximation $\hat{\phi}_0$ explicitly as follows. Assume $\operatorname{vol} Y = 1$.

(1.9) **Proposition.** *For a vector $v \in H^1_A$ with $|v|$ small, we have*

$$\hat{\phi}_0(v) = 2\left\{ \sum_{\alpha=1}^{h^1(L^2)} |Z_v^\alpha|^2 - \sum_{\beta=1}^{h^1(L^{-2})} |W_v^\beta|^2 \right\}.$$

Proof. We show $\hat{\phi}_0$ satisfies the system of differential equations

$$\begin{aligned} &\left.\frac{\partial \hat{\phi}_0}{\partial a_\alpha}\right|_v = 4 \operatorname{Re} Z_v^\alpha, \qquad \left.\frac{\partial \hat{\phi}_0}{\partial I a_\alpha}\right|_v = 4 \operatorname{Im} Z_v^\alpha, \qquad \alpha = 1, \dots, h^1(L^2); \\ &\left.\frac{\partial \hat{\phi}_0}{\partial b_\beta}\right|_v = -4 \operatorname{Re} W_v^\beta, \qquad \left.\frac{\partial \hat{\phi}_0}{\partial I b_\beta}\right|_v = -4 \operatorname{Im} W_v^\beta, \quad \beta = 1, \dots, h^1(L^{-2}). \end{aligned} \tag{1.10}$$

Then, as $\hat{\phi}_0(0) = 0$, it follows easily that

$$\begin{aligned}\hat{\phi}_0(v) &= 2\sum_\alpha\{(\operatorname{Re} Z_v^\alpha)^2 + (\operatorname{Im} Z_v^\alpha)^2\} - 2\sum_\beta\{(\operatorname{Re} W_v^\beta)^2 + (\operatorname{Im} W_v^\beta)^2\} \\ &= 2\left\{\sum_\alpha |Z_v^\alpha|^2 - \sum_\beta |W_v^\beta|^2\right\},\end{aligned}$$

as wished. To show (1.10) we check only

$$\left.\frac{\partial\hat{\phi}_0}{\partial a_1}\right|_v = 4\operatorname{Re} Z_v^1 \tag{1.11}$$

as the argument for other cases are similar. It is, however, just a routine matter of showing

$$\begin{aligned} d\hat{\phi}_0|_v(a_1) &= -\int_Y \operatorname{Tr}\left(\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}, d^+_{A+v}\begin{pmatrix} 0 & \varphi_1 \\ -\bar{\varphi}_1 & 0\end{pmatrix}\right)\wedge\omega_0 \\ &= \int_Y \operatorname{Tr}\left(\left[v, \begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}\right]\wedge\begin{pmatrix} 0 & \varphi_1 \\ -\bar{\varphi}_1 & 0\end{pmatrix}\right)\wedge\omega_0 \\ &= -2\int_Y \operatorname{Tr}\left(\begin{pmatrix} 0 & iZ_v^1\varphi_1 \\ i\overline{Z_v^1}\bar{\varphi}_1 & 0\end{pmatrix}\wedge\begin{pmatrix} 0 & \varphi_1 \\ -\bar{\varphi}_1 & 0\end{pmatrix}\right)\wedge\omega_0 \\ &= 2\int_Y (Z_v^1 + \overline{Z_v^1})i\varphi_1\wedge\bar{\varphi}_1\wedge\omega_0 = 4\operatorname{Re} Z_v^1.\end{aligned}$$

Now combining (1.8) and (1.9), we obtain

$$\hat{\phi}(v) = 2\left\{\sum_\alpha^{h^1(L^2)} |Z_v^\alpha|^2 - \sum_\beta^{h^1(L^{-2})} |W_v^\beta|^2\right\} + O(|v|^3), \tag{1.12}$$

which is the key result of this discussion. Using (1.12), one can deduce, in the case when both $h^1(L^2)$ and $h^1(L^{-2})$ are strictly positive, that the link of the reduction $[A]\in M_k(m_0)$ is a quotient

$$(S^{2h^1(L^2)-1}\times S^{2h^1(L^{-2})-1})/S^1$$

where S^1 acts diagonally on the spheres $S^{2h^1(L^2)-1}$ and $S^{2h^1(L^{-2})-1}$. Also by varying m_0 in a small path of metrics, one obtains a parametrized version of (1.12) that enables one to give an analytical proof of [M, Proposition (4.6)] concerning how a certain moduli space of stable 2-bundles over a complex quadric surface changes as the polarization varies. The details of showing these assertions are left to those interested.

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