PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 117, Number 1, January 1993

TOWARD A PRECISE SMOOTHNESS HYPOTHESIS IN SARD'S THEOREM

S. M. BATES

(Communicated by Charles Pugh)

ABSTRACT. The familiar $\mathbf{C}^k$ smoothness hypothesis in the Morse-Sard Theorem can be weakened to $\mathbf{C}^{k-1,1}$.

I. INTRODUCTION

Let $f: \mathbf{R}^n \to \mathbf{R}^m$ be a differentiable mapping. A question of fundamental importance in differential topology and dynamical systems concerns the measure of the critical value set of f, i.e., the image under f of those points $x \in \mathbf{R}^n$ such that $Df(x)$ is not surjective. The basic theorem regarding this problem is due to Morse and Sard (see [9, 10]):

The Morse-Sard Theorem. *Let $f: \mathbf{R}^n \to \mathbf{R}^m$ be a $\mathbf{C}^k$ map. If $k \geq \max\{n-m+1, 1\}$, then the set of critical values of f has Lebesgue m-measure zero.*

The necessity of the above differentiability requirement was established by Whitney [11], who constructed a $\mathbf{C}^1$ map $f: \mathbf{R}^2 \to \mathbf{R}$ not constant on a connected set of critical points (see also [1, 2, 4, 5]). Since the appearance of Whitney's example, investigations of the geometry of critical sets and critical values have led to various more precise formulations of the Morse-Sard Theorem (e.g., [3, 7, 8, 12]).

A map $f \in \mathbf{C}^p(\mathbf{R}^n, \mathbf{R}^m)$ is said to belong to $\mathbf{C}^{p,1}$ if $D^p f$ is locally Lipschitz on $\mathbf{R}^n$. One object of this paper is to prove

Theorem 1. *Let n, m be positive integers with $n > m$ and $k = n - m + 1$. If $f \in \mathbf{C}^{k-1,1}(\mathbf{R}^n, \mathbf{R}^m)$ then the set of critical values of f has m-measure zero.*

Thus the differentiability requirement in the classical Morse-Sard Theorem can be relaxed from $\mathbf{C}^k$ to $\mathbf{C}^{k-1,1}$.

For $\alpha \in (0, 1)$, recall that $f \in \mathbf{C}^p(\mathbf{R}^n, \mathbf{R}^m)$ is said to be of (Hölder) smoothness class $\mathbf{C}^{p,\alpha} = \mathbf{C}^{p+\alpha}$ if for every compact $B \subset \mathbf{R}^n$, there exists a real constant

Received by the editors May 24, 1991.
1991 *Mathematics Subject Classification.* Primary 58C25, 58C27; Secondary 26C35, 57R70.
Supported by an NSF graduate fellowship in Mathematics.

M such that

$$\|D^p f(x) - D^p f(y)\| \leq M|x-y|^\alpha$$

for all $x, y \in B$. A subset $E \subset \mathbf{R}^n$ is called a *set of* rank r *for* f provided rank $Df(x) \leq r$ for all $x \in E$. The central theorem of this paper is

Theorem 2. *Let n, m, and r be nonnegative integers satisfying $n > m > r$, and define $s = (n-r)/(m-r)$. If E is a set of rank r for $f\colon \mathbf{R}^n \to \mathbf{R}^m$ and either*

(a) *$s \in \mathbf{Z}^+$ and $f \in \mathbf{C}^{s-1,1}$ or*

(b) *$f \in \mathbf{C}^s$,*

then $f(E)$ has m-measure zero.

Note that by substituting $r = m-1$ into the statement of Theorem 2, one obtains Theorem 1.

We remark that Norton [7, 8] has proven Theorem 2 under more stringent smoothness requirements or, alternatively, under the assumption that E has n-measure zero. Theorem 2(a) is stronger than Federer's Theorem 3.4.3 [3], which implies the same conclusion for integer s provided $f \in \mathbf{C}^s$. Theorem 2(b) improves the estimates given by Yomdin's Theorem 5.3 [12], which concludes that the entropy dimension (and hence Hausdorff dimension) of $f(E)$ is at most m.

The degree to which Theorem 2 is itself sharp is illustrated by the following fact, proven by the author in [2].

Theorem 3. *For n, m, r, s as in Theorem 2, there exists a map $f\colon \mathbf{R}^n \to \mathbf{R}^m$ contained in $\mathbf{C}^t$ for all real $t < s$ and a subset E of rank r for f such that $f(E)$ contains an open set.*

Thus any further improvement of Theorem 2 and in particular of the Morse-Sard Theorem must distinguish between $\mathbf{C}^{p,1}$ and smoothness classes contained in $\mathbf{C}^t$ for all $t < p+1$.

Finally, we note that an application of Theorem 2 to singular mappings appears in [2].

We denote by l_m the Lebesgue outer measure on $\mathbf{R}^m$. A subset $E \subset \mathbf{R}^m$ is called m-null, resp. m-finite, provided $l_m(E) = 0$, resp. $l_m(E) < \infty$.

II. Proof of Theorem 2

A crucial observation is that the proof of Theorem 2 reduces to the case $r = 0$; accordingly, our goal in the next section is to prove

Lemma 1. *Let n, m be positive integers with $n > m$, and define $v = n/m$. If $E \subset \mathbf{R}^n$ is a set of rank 0 for $f\colon \mathbf{R}^n \to \mathbf{R}^m$ and either*

(a) *$v \in \mathbf{Z}^+$ and $f \in \mathbf{C}^{v-1,1}$ or*

(b) *$f \in \mathbf{C}^v$,*

then $f(E)$ is m-null.

Assuming Lemma 1 for the time being, we now derive Theorem 2(a) following Sard and Norton:

For $i = 0, 1, \ldots, r$, $r < m$, define $R_i = \{x \in E : \operatorname{rank} Df(x) = i\}$; we show $l_m f(R_i) = 0$ for each i.

(1) $l_m f(R_0) = 0$ by Lemma 1(a), since $v \leq s$ and f is $\mathbf{C}^{s-1,1}$.

(2) Fix $i \in \{1, \ldots, r\}$ and $p \in R_i$. It will suffice to find a neighborhood V of p such that $f(V \cap R_i)$ is m-null. By the $\mathbf{C}^{s-1,1}$ Inverse Function Theorem (see [7, 8]), there are coordinates in some neighborhood V of p such that

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_i, g(x_1, \ldots, x_n)),$$

where $g \in \mathbf{C}^{s-1,1}(\mathbf{R}^n, \mathbf{R}^{m-i})$. In these coordinates,

$$Df(y) = \begin{pmatrix} \mathrm{Id}_i & 0 \\ * & D(g\|x_1, \ldots, x_i) \end{pmatrix},$$

where Id_i is the identity matrix on $\mathbf{R}^i$, $y = (x_1, \ldots, x_n)$, and $g\|x_1, \ldots, x_i$: $\mathbf{R}^{n-i} \to \mathbf{R}^{m-i}$ is the mapping defined by $(x_{i+1}, \ldots, x_n) \mapsto g(x_1, \ldots, x_n)$. Note that $D(g\|x_1, \ldots, x_i)$ has rank 0 for all $y \in R_i$.

For $S \subset \mathbf{R}^n$ and $(x_1, \ldots, x_i) \in \mathbf{R}^i$, we denote by $S[x_1, \ldots, x_i]$ the set of points $(x_{i+1}, \ldots, x_n) \in \mathbf{R}^{n-i}$ such that $(x_1, \ldots, x_n) \in S$ (this is the "slice" of S through $(x_1, \ldots, x_i)$). Then $g\|x_1, \ldots, x_i$ maps the rank 0 set $(V \cap R_i)[x_1, \ldots, x_i]$ onto $(f(V \cap R_i))[x_1, \ldots, x_i]$.

Because $(n-i)/(m-i) \le (n-r)/(m-r) = s$ and $g\|x_1, \ldots, x_i$ is $\mathbf{C}^{s-1,1}$, it follows from Lemma 1(a) that every slice of $f(V \cap R_i)$ is $(m-i)$-null in $\mathbf{R}^{m-i}$. Applying Fubini's Theorem to $\mathbf{R}^i \times \mathbf{R}^{m-i}$, we conclude that $f(V \cap R_i)$ is m-null, and Theorem 2(a) follows.

To prove Theorem 2(b), one proceeds analogously, replacing $\mathbf{C}^{s-1,1}$ with $\mathbf{C}^s$ and Lemma 1(a) with Lemma 1(b) throughout the above argument.

III. Proof of Lemma 1

In order to prove Lemma 1, we require the following result due to Norton.

Lemma 2 (Norton). *Let n, m be positive integers with $n > m$, and define $v = n/m$. If E is an n-null set of rank 0 for $f: \mathbf{R}^n \to \mathbf{R}^m$ and either*

(a) *$v \in \mathbf{Z}^+$ and $f \in \mathbf{C}^{v-1,1}$, or*

(b) *$f \in \mathbf{C}^v$,*

then $f(E)$ is m-null.

The crux of our problem is thus to eliminate from Norton's Lemma the assumption that E is n-null; for this purpose, we need a further result of Norton:

The Generalized Morse Criticality Theorem (Norton). *Let n and p be positive integers, $t \ge 1$ a real number, $E \subset \mathbf{R}^n$.*

(a) *There are subsets E_j, $j = 0, 1, \ldots$, of E with $E = \bigcup E_j$ such that E_0 is countable and any $g \in \mathbf{C}^{p,1}(\mathbf{R}^n, \mathbf{R})$ critical on E satisfies, for each j,*

$$|g(x) - g(y)| \le M_j |x-y|^{p+1}$$

for all $x, y \in E_j$ and some $M_j > 1$.

(b) *There are subsets E'_j, $j = 0, 1, \ldots$, of E with $E = \bigcup E'_j$ such that E_0 is countable and any $g \in \mathbf{C}^t(\mathbf{R}^n, \mathbf{R})$ critical on E satisfies, for each j,*

$$|g(x) - g(y)| \le M_j |x-y|^t$$

for all $x, y \in E'_j$ and some $M_j > 1$.

Finally, recall that an element $x \in \mathbf{R}^n$ is called a *density point* of $E \subset \mathbf{R}^n$ if for any sequence of cubes $\{Q\}$ decreasing to x, the limit

$$\lim_{Q \searrow x} \frac{l_n(E \cap Q)}{l_n(Q)} = 1$$

holds. A familiar theorem from analysis asserts that almost all points of a measurable set are density points.

We now prove Lemma 1(a). Let n, m be positive integers, define $v = n/m$, and suppose $E \subset \mathbf{R}^n$ is a rank 0 subset for a map $f: \mathbf{R}^n \to \mathbf{R}^m$. Our objective is to show that $f(E)$ is m-null provided v is an integer and $f \in \mathbf{C}^{v-1,1}$.

Choose a decomposition $\{E_j\}$ of E as in part (a) of the Generalized Morse Theorem. Clearly we may suppose that each E_j is n-finite; we now demonstrate that $f(E_j)$ is m-null for all j:

(a) Since E_0 is countable, $f(E_0)$ is m-null.
(b) Fix $j \geq 1$ and a positive integer P.

(i) By part (a) of Norton's Lemma, we may assume that every point of E_j is a density point of E_j. Choose $x_0 \in E_j$, and let Q be a cube of edge λ containing x_0 and small enough that any cube $Q' \subset Q$ of edge $\lambda(2nP)^{-1}$ intersects E_j. If x, y are any two elements of $Q \cap E_j$, there evidently exists a covering of the line segment $\overline{xy}$ by at most $2nP$ such subcubes of Q. Consequently, there exists a sequence $\{x_i\}_{i=0}^{2nP} \subset E$ containing x, y and satisfying $|x_i - x_{i+1}| < \lambda/P$ for $i = 1, \dots, 2nP-1$. By part (a) of the Generalized Morse Theorem applied to each component function of f, there exists a constant $M > 1$ depending only on f and E_j such that

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x_1) - f(x_2)| + \cdots + |f(x_{2nP-1}) - f(x_{2nP})| \\ &\leq M(|x_1 - x_2|^v + \cdots + |x_{2nP-1} - x_{2nP}|^v) < 2nM\lambda^v P^{1-v}. \end{aligned}$$

Thus $l_m(f(Q \cap E_j)) \leq (2nM)^m \lambda^n P^{m(1-v)} = (2nM)^m P^{m(1-v)} l_n(Q)$.

(ii) Since for a given P the family of all cubes Q chosen as above for all $x_0 \in E_j$ evidently comprises a Vitali family for E_j, there exists a sequence $\{Q_l\}$ of such cubes satisfying $l_n(E_j \setminus \bigcup Q_l) = 0$ and $\sum l_n(Q_l) < 2l_n(E_j)$.

Applying part (a) of Norton's Lemma once again, it follows that

$$\begin{aligned} l_m(f(E_j)) &\leq \sum l_m(f(Q_l \cap E_j)) \leq (2nM)^m P^{m(1-v)} \sum l_n(Q_l) \\ &< 2(2nM)^m P^{m(1-v)} l_n(E_j). \end{aligned}$$

Since P may be chosen arbitrarily large, $f(E_j)$ is m-null, and the assertion follows.

To prove Lemma 1(b), choose a decomposition $\{E_j'\}$ of E as in part (b) of the Generalized Morse Theorem, and proceed analogously.

Acknowledgments

I am happy to thank R. Kaufman for his suggestions in [6] and C. Pugh for helpful criticisms of the manuscript.

REFERENCES

1. S. M. Bates, *On the image size of singular maps*. I, Proc. Amer. Math. Soc. **114** (1992), 699–705.
2. ———, *On the image size of singular maps*. II, preprint.
3. H. Federer, *Geometric measure theory*, Grundlehren Math. Wiss., vol. 153, Springer-Verlag, New York, 1969.
4. E. L. Grinberg, *On the smoothness hypothesis in Sard's Theorem*, Amer. Math. Monthly **47** (1985), 299–304.
5. R. Kaufman, *A singular map of a cube onto a square*, J. Differential Geom. **14** (1979), 593–594.
6. ———, private communication, August 1990.
7. A. Norton, *A critical set with nonnull image has large Hausdorff dimension*, Trans. Amer. Math. Soc. **296** (1986), 367–376.
8. ——— *The fractal geometry of critical sets with nonnull image and the differentiability of functions*, Ph.D. Thesis, Univ. of California, Berkeley, 1987.
9. A. Sard, *The measure of the critical values of differentiable maps*, Bull. Amer. Math. Soc. **48** (1942), 883–890.
10. S. Sternberg, *Lectures on differential geometry*, Prentice-Hall, Englewood Cliffs, NJ, 1964.
11. H. Whitney, *A function not constant on a connected set of critical points*, Duke Math J. **1** (1935), 514–517.
12. Y. Yomdin, *The geometry of critical and near-critical values of differentiable mappings*, Math. Ann. **264** (1983), 495–515.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT BERKELEY, BERKELEY, CALIFORNIA 94720