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MAJORIZATION AND DOMINATION IN THE BERGMAN SPACE

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ABSTRACT. Let f and g be functions analytic on the unit disk and let $\|\cdot\|$ denote the Bergman norm. Conditions are identified under which there exists an absolute constant c, with $0 < c < 1$, such that the relationship $|g(z)| \leq |f(z)|$ $(c \leq |z| < 1)$ will imply $\|g\| \leq \|f\|$.

1. INTRODUCTION

Let $\mathbb{C}$ denote the complex plane, $\mathbb{D}$ the open unit disk, and $L^2(\mathbb{D})$ the Hilbert space of all measurable functions $f\colon \mathbb{D} \to \mathbb{C}$ with

$$\|f\|^2 = \frac{1}{\pi}\int_{\mathbb{D}} |f|^2\, dm < \infty,$$

where dm denotes the Lebesgue area measure. The Bergman space A_2 is defined to be the subspace of $L^2(\mathbb{D})$ consisting of functions analytic on $\mathbb{D}$.

Let f and g be analytic on $\mathbb{D}$. We say g is *majorized* by f on a region $R \subseteq \mathbb{D}$ if $|g(z)| \leq |f(z)|$ for all $z \in R$. By the positivity of dm, if g is majorized by f on $\mathbb{D}$ then certainly

$$\|g\| \leq \|f\|. \tag{1}$$

What other cases of majorization will imply (1)? In particular, when does majorization on an annulus imply (1)? That is, we investigate the existence of an absolute constant c, with $0 < c < 1$, such that if

$$|g(z)| \leq |f(z)| \qquad (c \leq |z| < 1) \tag{2}$$

then

$$\|g\| \leq \|f\|. \tag{3}$$

In the case that either function is a monomial z^n, it has been shown (see [3]) that (2) implies (3) for any $c \leq 1/\sqrt{3}$.

For arbitrary f and g analytic on $\mathbb{D}$, let Z_f and Z_g denote the zero sets (counting multiplicity) of f and g, respectively. If $Z_f \backslash Z_g$ is empty, then (2) implies (3) for any $c \in (0, 1)$, by the classical maximum principle.

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Furthermore, it is known (see [1]) that when $Z_g \backslash Z_f$ is empty (2) will again imply (3) for $c \leq 1/(2e^2)$.

We seek conditions under which majorization on an annulus of inner radius $c \in (0, 1)$ will imply (3), without requiring either relative complementary zero set to be empty. We will show that such a c exists when $Z_g \backslash Z_f$ and $Z_f \backslash Z_g$ are separated by an annulus.

2. Preliminary results

First we present some definitions and preliminary facts that will be appealed to throughout this article (see [1, 2]). Let H^∞ be the space of all bounded analytic functions on $\mathbb{D}$, with

$$\|h\|_\infty = \sup\{|h(z)| \,|\, z \in \mathbb{D}\}.$$

For $G, F \in L^2(\mathbb{D})$, we say that G is *dominated* by F if $\|Gh\| \leq \|Fh\|$ for all $h \in H^\infty$, and we write $G \prec F$. It follows that if $G \prec F$ and $G, F \in L^\infty(\mathbb{D})$, then $\|Gh\| \leq \|Fh\|$ for all $h \in A_2$. The following properties are direct consequences of the definition of domination.

Property 1. *If $G, F \in L^2(\mathbb{D})$ and $G \prec F$, then $G \circ \phi \prec F \circ \phi$ for all Möbius transformations ϕ on $\mathbb{D}$.*

Property 2. *If $G_i, F_i \in H^\infty$ and $G_i \prec F_i$ for $i = 1, \ldots, n$, then*

$$(G_1 G_2, \ldots, G_n) \prec (F_1 F_2, \ldots, F_n).$$

We denote $B_a(z) = \bar{a}(a-z)/[|a|(1-\bar{a}z)]$ for $a \in \mathbb{D}$, $a \neq 0$, and $B_0(z) = -z$.

Proposition 1. *Let $a \in \mathbb{D}$ and $\gamma > 0$. Then*

$$|B_a|^{\gamma((1+|a|)/(1-|a|))} \prec |z|^\gamma, \tag{4}$$

and

$$\exp\{-2\gamma(1+z)/(1-z)\} \prec |z|^\gamma. \tag{5}$$

Corollary 1. *Let $a \in \mathbb{D}$ and define $\alpha = (1-|a|)/(1+|a|)$, $\beta = (1+|a|)/(1-|a|)$. Then $|z|^\beta \prec B_a \prec |z|^\alpha$.*

(Proposition 1 is proved in [2] while Corollary 1 follows from (4) and Property 1.)

Corollary 2. *Let $B(z) = \prod_{k=1}^N B_{a_k}(z)$, $|a_k| < 1$, and let*

$$\beta = \sum_{k=1}^N \left(\left[\frac{1+|a_k|}{1-|a_k|} \right] + 1 \right),$$

where $N < \infty$ and $[\cdot]$ denotes the greatest integer function. Then $|z|^\beta \prec B$.

Proof. By Corollary 1, we have for each k that $|z|^{\beta_k} \prec B_{a_k}$ where $\beta_k = (1+|a_k|)/(1-|a_k|)$. Since $\beta_k \leq ([\beta_k]+1)$, we have

$$z^{([\beta_k]+1)} \prec B_{a_k}. \tag{6}$$

Since both sides of (6) are in H^∞, we can apply Property 2 for $k = 1, \ldots, N$, which yields $|z|^\beta \prec B$. This proves the corollary. □

Proposition 2 (see [1]). *Suppose* $G \in H^\infty$, $\|G\|_\infty \leq 1$, *and* $G(z) \neq 0$ *for all* $z \in \mathbb{D}$. *If* $|G(0)| \leq e^{-2\gamma}$ *for* $\gamma > 0$, *then* $G \prec |z|^\gamma$.

Proof. We have

$$G(z) = \lambda \exp\left\{-\int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta)\right\},$$

where $d\mu$ is a positive Borel measure on $\partial\mathbb{D}$, with $\mu(\partial\mathbb{D}) = -\log|G(0)| \geq 2\gamma$, and $|\lambda| = 1$. Let $\eta = \mu(\partial\mathbb{D})$. Using the generalized arithmetic-geometric mean inequality combined with the Fubini Theorem and (5), we obtain for all $h \in A_2$

$$\begin{aligned}
\int_{\mathbb{D}} |Gh|^2\, dm &= \int_{\mathbb{D}} \exp\left\{-2\eta \int_0^{2\pi} \operatorname{Re} \frac{e^{it} + z}{e^{it} - z} \cdot \frac{d\mu(t)}{\eta}\right\} |h(z)|^2\, dm_z \\
&\leq \int_{\mathbb{D}} \left[\int_0^{2\pi} \left|\exp\left\{\frac{-2\eta(e^{it} + z)}{e^{it} - z}\right\}\right| \frac{d\mu(t)}{\eta}\right] |h(z)|^2\, dm_z \\
&= \int_0^{2\pi} \int_{\mathbb{D}} \left|\exp\left\{\frac{-\eta(e^{it} + z)}{e^{it} - z}\right\} h(z)\right|^2 dm_z \frac{d\mu(t)}{\eta} \\
&\leq \int_{\mathbb{D}} |z|^\eta |h(z)|^2\, dm \leq \int_{\mathbb{D}} |z|^{2\gamma} |h(z)|^2\, dm. \quad \square
\end{aligned}$$

3. Main results

Theorem 1. *Let* $G \in H^\infty$ *be such that* $\|G\|_\infty \leq 1$ *and* $G(z) \neq 0$ *for all* $|z| < c$ *for some* $c \in (0, 1)$. *If* $|G(0)| \leq (c^{(1+c)/(1-c)})^\gamma$ *for some* $\gamma > 0$, *then* $G \prec |z|^\gamma$.

(It is interesting to note that Proposition 2 is the limiting case of Theorem 1 since $c^{(1+c)/(1-c)} \to e^{-2}$ as $c \to 1^-$.) We postpone the proof of Theorem 1 until the end of the article. Once this theorem is proved, we can obtain the following results.

Theorem 2. *Let* $B(z) = \prod_{k=1}^N B_{a_k}(z)$, *where* $|a_k| \leq c < \frac{1}{3}$. *Suppose* $G \in H^\infty$, $\|G\|_\infty \leq 1$, *and* $G(z) \neq 0$ *for all* $|z| < d$, *with* $d > c$. *If*

$$|G(0)| \leq (d^{(1+d)/(1-d)})^{2N},$$

then $G \prec B$.

Proof. We have that $G \prec |z|^{2N}$, by Theorem 1. Since $|a_k| < \frac{1}{3}$, it follows that

$$\beta = \sum_{k=1}^{N} \left(\left[\frac{1 + |a_k|}{1 - |a_k|}\right] + 1\right) = 2N.$$

Thus, $|z|^{2N} \prec B$, by Corollary 2. Therefore, $G \prec |z|^{2N} \prec B$, which proves the theorem. $\square$

Theorem 3. *Let* $f, g \in A_2$. *There exists an absolute constant* c_0, *with* $0 < c_0 < 1$, *such that for any* $c < c_0$, *if*

(i) $|g(z)| \leq |f(z)|$ $(c \leq |z| < 1)$ *and*
(ii) $Z_g \backslash Z_f \subset \{z \in \mathbb{D} | c^{1/3} < |z| < 1\}$,

then $g \prec f$.

Proof. Let B be the finite Blaschke product

$$B = \prod_{k=1}^{N} B_{a_k}, \qquad \text{where } \{a_k\}_{k=1}^{N} = Z_f \backslash Z_g.$$

Note that (i) implies $|a_k| < c$ for $k = 1, \ldots, N$. Consider the function $G = gB/f$. It follows that $G \in H^\infty$, $\|G\|_\infty \le 1$, and $G(z) \neq 0$ for $|z| < c^{1/3}$. Also, on $|z| = c$, we have

$$|G(z)| \le |B(z)| \le (2c)^N.$$

The classical maximal principle implies that $|G(0)| \le (2c)^N$. Letting $d = c^{1/3}$, it follows that $2c \le d^{2(1+d)/(1-d)}$ for $c < 0.0021 = c_0$. Thus,

$$|G(0)| \le (d^{(1+d)/(1-d)})^{2N}$$

and $G(z) \neq 0$ for all $|z| < d$. Applying Theorem 2, we have $G \prec B$ or $\|Gh\| \le \|Bh\|$ for all $h \in A_2$. Taking $h = h_1 f/B$, with $h_1 \in H^\infty$, we have $g \prec f$. □

Corollary 3. *Let f and g be analytic on $\mathbb{D}$. There exists an absolute constant c_0, with $0 < c_0 < 1$, such that for any $c < c_0$, if*

(i) $|g(z)| \le |f(z)|$ $(c \le |z| < 1)$ *and*
(ii) $Z_g \backslash Z_f \subset \{z \in \mathbb{D} | c^{1/3} < |z| < 1\}$,

then $\|g\| < \|f\|$.

Proof. If $\|f\| = \infty$, then there is nothing to prove. Thus we can assume that $f \in A_2$, which implies $g \in A_2$ by (i). Applying Theorem 3, we have $g \prec f$ and, in particular, $\|g\| \le \|f\|$. □

Notice that (i) and (ii) together imply that $Z_g \backslash Z_f$ is separated from $Z_f \backslash Z_g$ by the annulus $\{z \in \mathbb{D} | c \le |z| \le c^{1/3}\}$.

Proof of Theorem 1. Let $F = G/B$ where $B = \prod_{k=1}^{M} B_{a_k}$ and $\{a_k\}_{k=1}^{M} = Z_G$ $(M \le \infty)$. Applying (4), we have for $\lambda, \gamma > 0$

$$|B_{a_k}|^{\lambda\gamma\alpha_k} \prec |z|^{\lambda\gamma} \tag{7}$$

where $\alpha_k = (1 + |a_k|)/(1 - |a_k|)$. Since F has no zeros in $\mathbb{D}$ and $\|F\|_\infty \le 1$, we can define the analytic function $\widehat{F} = F^{2\lambda\gamma/\log(1/|F(0)|)}$ that satisfies $\|\widehat{F}\|_\infty \le 1$ and $|\widehat{F}(0)| = e^{-2\lambda\gamma}$. Thus, we can apply Proposition 2 to $\widehat{F}$ yielding

$$\widehat{F} \prec |z|^{\lambda\gamma}. \tag{8}$$

Now let $h \in H^\infty$. We have

$$\int_{\mathbb{D}} |Gh|^2 \, dm = \int_{\mathbb{D}} \left| \prod_k B_{\alpha_k}(z) \right|^2 |F(z)h(z)|^2 \, dm \tag{9}$$

$$= \int_{\mathbb{D}} \prod_k (|B_{a_k}(z)|^{2\lambda\gamma\alpha_k})^{1/(\lambda\gamma\alpha_k)} (|\widehat{F}(z)|^2)^{(\log 1/|F(0)|)/2\lambda\gamma} |h(z)|^2 \, dm, \tag{10}$$

where we choose

$$\lambda = \frac{1}{\gamma} \left[\sum_k \left(\frac{1}{\alpha_k} \right) + \frac{1}{2} \log \frac{1}{|F(0)|} \right].$$

For this λ, we can apply the arithmetic-geometric mean inequality to (10) to obtain

$$\int_{\mathbb{D}} |Gh|^2\,dm \le \sum_k \frac{1}{\lambda\gamma\alpha_k} \int_{\mathbb{D}} (|B_{a_k}|^{\lambda\gamma\alpha_k}|h(z)|)^2\,dm + \frac{\log(1/|F(0)|)}{2\lambda\gamma} \int_{\mathbb{D}} |\widehat{F}(z)h(z)|^2\,dm.$$

Applying (7) and (8), it follows that

$$\begin{aligned}\int_{\mathbb{D}} |Gh|^2\,dm &\le \left[\sum_k \left(\frac{1}{\lambda\gamma\alpha_k}\right) + \frac{\log(1/|F(0)|)}{2\lambda\gamma}\right] \int_{\mathbb{D}} |z|^{2\lambda\gamma}|h(z)|^2\,dm \\ &= \int_{\mathbb{D}} |z|^{2\lambda\gamma}|h(z)|^2\,dm,\end{aligned}$$

by the above choice of λ. We will show that $\lambda \ge 1$. Once this is done, it follows that

$$\int_{\mathbb{D}} |Gh|^2\,dm \le \int_{\mathbb{D}} |z|^{2\gamma}|h(z)|^2\,dm,$$

which is the desired result. To see that $\lambda \ge 1$, first observe that

$$|F(0)|\prod_k |a_k| = |F(0)B(0)| = |G(0)| \le c^{\gamma(1+c)/(1-c)}, \tag{11}$$

where the last inequality follows by hypothesis. This implies

$$\frac{1+c}{1-c}\log\frac{1}{c} \le \frac{1}{\gamma}\sum_k \left(\log\frac{1}{|a_k|}\right) + \frac{1}{\gamma}\log\frac{1}{|F(0)|}. \tag{12}$$

It can be shown that the function $\phi(r) = \frac{1+r}{1-r}\log\frac{1}{r}$ is decreasing and $\phi(r) > 2$ on $(0, 1)$. We have, by assumption, that $|a_k| > c$ and thus, $\phi(|a_k|) \le \phi(c)$ for all k. This yields

$$\log\frac{1}{|a_k|} \le \left(\frac{1-|a_k|}{1+|a_k|}\right)\frac{1+c}{1-c}\log\frac{1}{c} = \frac{1}{\alpha_k}\left(\frac{1+c}{1-c}\log\frac{1}{c}\right).$$

This, together with (12), implies

$$\frac{1+c}{1-c}\log\frac{1}{c} \le \frac{1}{\gamma}\left[\sum_k \left(\frac{1}{\alpha_k}\right)\frac{1+c}{1-c}\log\frac{1}{c} + \log\frac{1}{|F(0)|}\right]$$

or

$$\begin{aligned}1 &\le \frac{1}{\gamma}\left[\sum_k \left(\frac{1}{\alpha_k}\right) + \frac{2(1-c)}{(1+c)\log 1/c}\cdot\frac{1}{2}\log\frac{1}{|F(0)|}\right] \\ &\le \frac{1}{\gamma}\left[\sum_k \left(\frac{1}{\alpha_k}\right) + \frac{1}{2}\log\frac{1}{|F(0)|}\right] = \lambda,\end{aligned}$$

where the last inequality is a consequence of $\phi(c) > 2$. This completes the proof of the theorem. □

References

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