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A GENERALIZATION OF THE PUNCTURED NEIGHBORHOOD THEOREM

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ABSTRACT. If $T \in \mathscr{L}(X)$ is regular on a Banach space X, with finite- dimensional intersection $T^{-1}(0) \cap T(X)$, and if S, S' are invertible, commute with T and have sufficiently small norm, then $\dim(T-S')^{-1}(0) = \dim(T-S)^{-1}(0)$ and $\dim X/(T-S')X = \dim X/(T-S)X$.

In [5], Lee proved that if T is a regular operator with some finite-dimensional intersection property on a Banach space and if 0 is the boundary of the spectrum of T, then 0 is an isolated point of the spectrum of T.

In this note we derive a generalization of the punctured neighborhood theorem and then strengthen the above result.

Throughout this note suppose X and Y are complex Banach spaces, write $\mathscr{L}(X, Y)$ for the set of bounded linear operators from X to Y, and abbreviate $\mathscr{L}(X, X)$ to $\mathscr{L}(X)$. If $T \in \mathscr{L}(X)$ then we write $\sigma(T)$ for the spectrum of T. If K is a compact subset of the complex plane $\mathbf{C}$, write ∂K and $\operatorname{iso}(K)$, respectively, for the topological boundary points and the isolated points of K.

We recall that $T \in \mathscr{L}(X, Y)$ is said to be *bounded below* if there is $k > 0$ for which $\|x\| \le k\|Tx\|$ for all $x \in X$ and is said to be *regular* if there is $T' \in \mathscr{L}(Y, X)$ for which $T = TT'T$. It is known that T is regular if and only if $T(X)$ is closed and both $T^{-1}(0)$ and $T(X)$ are complemented and that

$$(0.1) \qquad T \text{ regular and one-one } \Rightarrow T \text{ bounded below } \Rightarrow T(X) \text{ closed}$$

(cf. [2, 3]). Recall, also, that $T \in \mathscr{L}(X, Y)$ is said to be *Fredholm* if $T^{-1}(0)$ and $Y/T(X)$ are finite dimensional. If $T \in \mathscr{L}(X, Y)$ is Fredholm then the *index* of T is defined by

$$\operatorname{index}(T) = \dim T^{-1}(0) - \dim Y/T(X).$$

If $T \in \mathscr{L}(X)$ then the *hyperrange* of T is the subspace

$$T^{\infty}(X) = \bigcap_{n=1}^{\infty} T^n(X).$$

We begin with a modification of [2, Theorem 7.8.3]:

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Lemma 1. *Let X be a normed space and $T \in \mathscr{L}(X)$. If the intersection $T^{-1}(0) \cap T^k(X)$ is finite dimensional for some k then*

$$T(T^\infty(X)) = T^\infty(X). \tag{1.1}$$

If $S \in \mathscr{L}(X)$ is invertible and commutes with T, then

$$(T-S)^{-1}(0) \subseteq T^\infty(X). \tag{1.2}$$

Proof. The proof of equality (1.1) is taken straight from a slight modification of the proof of [2, (7.8.3.2)], which works with the stronger assumption $\dim T^{-1}(0) < \infty$. The inclusion (1.2) is just the inclusion [2, (7.8.3.4)].

Our main theorem is a generalization of the "punctured neighborhood theorem."

Theorem 2. *If $T \in \mathscr{L}(X)$ is regular on a Banach space X, with finite-dimensional intersection $T^{-1}(0) \cap T(X)$, and if S, S' are invertible, commute with T and have sufficiently small norm, then*

$$\dim(T-S')^{-1}(0) = \dim(T-S)^{-1}(0) \tag{2.1}$$

and

$$\dim X/(T-S')X = \dim X/(T-S)X. \tag{2.2}$$

Proof. Suppose $T \in \mathscr{L}(X)$ is regular and $T^{-1}(0) \cap T(X)$ is finite dimensional. We begin by showing $T^\infty(X)$ is complete. To do this, define $S_1 : X/T^{-1}(0) \to X$ by setting

$$S_1(x + T^{-1}(0)) = Tx \in X \quad \text{for each } x \in X.$$

Then, by (0.1) S_1 is bounded below. Our assumption also gives (with the aid of [5, Lemma 1]) that the subspace $T(X) + T^{-1}(0)$ is closed in X. Further, we can find a closed subspace $W \subseteq T(X)$ for which

$$T(X) + T^{-1}(0) = W + T^{-1}(0) \quad \text{and} \quad T(X) = W \oplus (T^{-1}(0) \cap T(X)).$$

We can then regard $W + T^{-1}(0) = \{w + T^{-1}(0) : w \in W\}$ as a closed subspace of $X/T^{-1}(0)$. If we define $S_2 : W + T^{-1}(0) \to X$ by setting

$$S_2(w + T^{-1}(0)) = Tw \in X \quad \text{for each } w \in W,$$

then S_2 is also bounded below (see [2, (3.11.1.2)]). Since $W + T^{-1}(0)$ is complete, it follows from (0.1) that $S_2(W + T^{-1}(0)) = T(W) = T^2(X)$ is closed in X. Inductively, we have that $T^n(X)$ is closed in X for each $n \in N$, hence so is $T^\infty(X)$; therefore, $T^\infty(X)$ is complete. We write $U^\wedge : T^\infty(X) \to T^\infty(X)$ for the operator induced by $U \in \operatorname{comm}(T)$, where $\operatorname{comm}(T)$ is the "commutant" of T in $\mathscr{L}(X)$. Then, since $(T^\wedge)^{-1}(0) = T^{-1}(0) \cap T^\infty(X) \subseteq T^{-1}(0) \cap T(X)$ and, by (1.1), $T^\wedge$ is onto, it follows that $T^\wedge$ is Fredholm. If S has sufficiently small norm then $(T-S)^\wedge$ is also Fredholm because the Fredholm operators on a Banach space form an open set. We now claim that

$$\dim(T-S)^{-1}(0) = \dim(T-S)^{\wedge^{-1}}(0) = \operatorname{index}(T-S)^\wedge = \operatorname{index}(T^\wedge). \tag{2.3}$$

The first equality comes from (1.2), the second equality comes from the fact that, by the first equality and (1.1), $(T-S)^\wedge$ is onto, and the third equality comes from the continuity of the Fredholm index. Since the right-hand side of

(2.3) is independent of S, equality (2.1) follows. Also applying the "classical" punctured neighborhood theorem of $T - S$ gives the equality (2.2).

Our proof of Theorem 2 closely follows the original argument of Harte [2, Theorem 7.8.4], which assumes T is Fredholm.

The following result is an improvement of [5, Theorem 2]:

Corollary 3. *If $T \in \mathscr{L}(X)$ is regular on a Banach space X, with finite-dimensional intersection $T^{-1}(0) \cap T(X)$, then there is implication*

$$0 \in \partial\sigma(T) \Rightarrow 0 \in \operatorname{iso}\sigma(T). \tag{3.1}$$

Proof. Apply Theorem 2 to $T - S$ with $S = \mu I$ and $0 < |\mu| < \varepsilon$; then $\dim(T - \lambda I)^{-1}(0)$ and $\dim X/(T - \lambda I)X$ are constant on a punctured neighborhood of 0. If $0 \in \partial\sigma(T)$ then it follows that for some θ with $0 < \theta < \varepsilon$, $\dim(T - \lambda I)^{-1}(0) = \dim X/(T - \lambda I)X = 0$ for $0 < |\lambda| < \theta$, which says that $0 \in \operatorname{iso}\sigma(T)$.

Recall that $T \in \mathscr{L}(X)$ is said to be *relatively almost open* if its truncation $\widehat{T}: X \to T(X)$ is almost open (cf. [2, 4]). If $T \in \mathscr{L}(X)$ for a Banach space X then by the open mapping theorem we have

$$T \text{ relatively almost open } \Leftrightarrow T(X) \text{ closed}. \tag{3.2}$$

In the context of a Hilbert space we can simplify Corollary 3. In a sense, the following result is an improvement of [6, Theorem 1].

Corollary 4. *If X is a Hilbert space and $T \in \mathscr{L}(X)$ is relatively almost open, with finite-dimensional intersection $T^{-1}(0) \cap T(X)$, then there is implication*

$$0 \in \partial\sigma(T) \Rightarrow 0 \in \operatorname{iso}\sigma(T). \tag{4.1}$$

Proof. If $T \in \mathscr{L}(X)$ for a Hilbert space X then both $T^{-1}(0)$ and $\operatorname{cl} T(X)$ are always complemented; thus we have

$$T \text{ regular } \Leftrightarrow T(X) \text{ closed}; \tag{4.2}$$

therefore, (3.1) together with (3.2) and (4.2) gives (4.1).

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