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HOLOMORPHIC MAPPINGS INTO TEICHMÜLLER SPACES

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ABSTRACT. Let $T(F)$ be the Teichmüller space of a finitely generated Fuchsian group F of the first kind. We shall show that there exists no nontrivial family of holomorphic proper mappings of the unit disc into $T(F)$ with a complex analytic parameter. Using this result, we shall investigate a family of quasi-conformal mappings with a complex analytic parameter whose Beltrami differentials vanish on a fixed set of positive area.

1. INTRODUCTION

Let F be a finitely generated Fuchsian group of the first kind. The Teichmüller space $T(F)$ of F is identified, via the Bers embedding, with a bounded domain in $\mathbf{C}^m$, where m is the dimension of $T(F)$. Hence a holomorphic mapping of the unit disc Δ into $T(F)$ is regarded as a finite system of bounded holomorphic functions on Δ. Shiga [Sh1, Sh2] applied theorems on bounded holomorphic functions to such a mapping to prove several results on Teichmüller spaces. On the other hand, a lot of properties of $T(F)$ and its boundary have been pointed out by many researchers. Some of these results, for example Lemmas 1, 2, 3 in §3, indicate that the boundary is complicated. With these properties, the Fatou's theorem and the Riesz' theorem about bounded analytic functions, Imayosi-Shiga [IS] gave an analytic proof of the finiteness theorem of holomorphic families over Riemann surfaces of finite type.

In this paper, we also utilize the properties of the boundary of $T(F)$ and theorems on bounded analytic functions to show certain rigidity properties of holomorphic proper mappings of the unit disk Δ into $T(F)$ (actually, we deal with holomorphic mappings under some weaker condition). Namely, there exists no nontrivial family of holomorphic proper mappings of Δ into $T(F)$ with a complex analytic parameter. As a corollary we shall show that $T(F)$ is not holomorphically equivalent to a product of complex manifolds. We shall also investigate, as a corollary, some families of quasi-conformal mappings of Riemann surfaces.

2. NOTATION

In this section we fix our notation and recall some known results. We refer to Kra [K2] and Lehto [L] for more details and references.

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Throughout this paper, Δ denotes the unit disk and H denotes the upper half plane and L denotes the lower half plane. In this paper, a quasi-conformal mapping is a quasi-conformal homeomorphism of the Riemann sphere $\widehat{C}$. Let F be a finitely generated Fuchsian group of the first kind acting on the upper half plane H. F is said to be of signature $(p, n; \alpha_1, \dots, \alpha_n)$ (or simply, (p, n)) if the Riemann surface H/F has genus p and has n special points with ramification number $\alpha_1, \dots, \alpha_n$. Here we consider a puncture of H/F as a special point of ramification number ∞. An essentially bounded measurable function on $\widehat{C}$ satisfying

$$\mu \circ g \left(\overline{g'}/g'\right) = \mu \quad \text{a.e.}$$

for all $g \in F$ is called a *Beltrami differential for* F.

Let $B(F)$ denote the space of Beltrami differentials for F. The space $B(F)$ is a Banach space with supremum norm. Let $B_1(F)$ denote the unit ball of $B(F)$. For F-invariant set U, $B(F, U)$ denotes the space of Beltrami differentials for F that vanish outside of U and $B_1(F, U)$ denotes the unit ball of $B(F, U)$. For each $\mu \in B_1(F)$, w^μ denotes the quasi-conformal homeomorphism of $\widehat{C}$ with Beltrami differential μ normalized so that w^μ fixes 0, 1, and ∞.

A quasi-conformal homeomorphism w is called *compatible with* F provided

$$w \circ g \circ w^{-1} \in \mathrm{PSL}(2, C)$$

for all $g \in F$. A quasi-conformal homeomorphism w is compatible with F if and only if its Beltrami differential belongs to $B_1(F)$. Two Beltrami differentials μ_1 and μ_2 are called *equivalent* if w^{μ_1} and w^{μ_2} induce the same group isomorphism of F into $\mathrm{PSL}(2, C)$; that is,

$$w^{\mu_1} \circ g \circ (w^{\mu_1})^{-1} = w^{\mu_2} \circ g \circ (w^{\mu_2})^{-1}$$

for all $g \in F$. The *Teichmüller space* $T(F)$ of the Fuchsian group F is the set of all equivalence classes that are determined by elements of $B_1(F, H)$. Let μ_1 and μ_2 be elements of $B_1(F, H)$. Set

$$\hat{\mu}_i(z) = \begin{cases} \mu_i(z) & z \in H, \\ \overline{\mu}_i(\overline{z}) & z \in L, \end{cases} \qquad i = 1, 2.$$

The Beltrami differentials μ_1 and μ_2 determine the same group isomorphism if and only if $\hat{\mu}_1$ and $\hat{\mu}_2$ determine the same group isomorphism.

Now we introduce the *Bers embedding* of $T(F)$ into C^{3p-3+n}. Let $A_2(F, L)$ be the space of holomorphic functions on the lower half plane L satisfying

$$\phi \circ g \times (g')^2 = \phi \quad \forall g \in F$$

and

$$\sup\{|(\mathrm{Im}(z))^2 \phi(z)|\,;\, z \in L\} < \infty.$$

The space $A_2(F, L)$ is a $(3p - 3 + n)$-dimensional complex linear space, hence equivalent to C^{3p-3+n}. Set

$$\widehat{\Phi}(\mu) = S_{w^\mu|L}$$

where $\mu \in B_1(F, H)$ and $S_{w^\mu|L}$ stands for the Schwarzian derivative of $w^\mu|L$. Then $\widehat{\Phi}(\mu)$ depends only on the equivalence class $[\mu] \in T(F)$ and $\widehat{\Phi}(\mu)$ belongs to $A_2(F, L)$. Thus the mapping $\widehat{\Phi}\colon B_1(F, H) \to A_2(F, L)$ induces a mapping $\Phi\colon T(F) \to A_2(F, L)$. The mapping Φ is a holomorphic injection and gives an embedding (called the Bers embedding) of $T(F)$ onto a bounded domain of $A_2(F, H)$. We identify $T(F)$ and $\Phi(T(F))$, and regard $T(F)$ as a bounded domain in C^{3p-3+n} .

We describe the boundary $\partial T(F)$ of $T(F)$ briefly (see Bers [B1], Maskit [M] for details). For each element $\phi \in \overline{T}(F)$ there exists a univalent function W_ϕ of the lower half plane L such that $S_W = \phi$ and that

$$W_\phi(z) = (z+i)^{-1} + O(|z+i|), \quad \text{near } z = -i.$$

Since $\phi \in \overline{T}(F) \subset A_2(F, L)$,

$$S_{W\circ g} = S_W = \phi,$$

for each element g of F. Hence there exists an element $\theta_\phi(g)$ of $\mathrm{PSL}(2, C)$ such that

$$\theta_\phi(g) \circ W_\phi = W_\phi \circ g.$$

The correspondence $g \mapsto \theta_\phi(g)$ is an isomorphism of F into $\mathrm{PSL}(2, C)$. It is known that if $\phi \in T(F)$ then $\theta_\phi(F)$ is a quasi-fuchsian group and that if $\phi \in \partial T(F)$, then $\theta_\phi(F)$ is a Kleinian group with exactly one invariant component $W_\phi(L)$. The Kleinian group $\theta_\phi(F)$ is called a *cusp* provided there exists a hyperbolic element $g \in F$ such that $\theta_\phi(g)$ is parabolic. The Kleinian group $\theta_\phi(F)$ with $\phi \in \partial T(F)$ is called a *regular b-group* if the Poincaré area of the Riemann surface $W_\phi(L)/\theta_\phi(F)$ is half of that of $\Omega(\theta_\phi(F))/\theta_\phi(F)$. The Kleinian group $\theta_\phi(F)$ is called a *totally degenerate b-group* if $W_\phi(L)$ is a dense open subset of $\widehat{C}$. It is known that if $\phi \in \partial T(F)$ and $\theta_\phi(F)$ is not a cusp, then $\theta_\phi(F)$ is a totally degenerate b-group.

For a quasi-conformal selfmapping w of $\widehat{C}$ with Beltami differential μ, the maximal dilatation $K(w)$ is given by

$$K(w) = \frac{1+||\mu||_\infty}{1-||\mu||_\infty}.$$

The *Teichmüller distance* d on $T(F)$ is defined by

$$d([\mu_0], [\nu_0]) = \inf\{\tfrac{1}{2}\log K(w^\mu \circ (w^\nu)^{-1})\,;\, \mu, \nu \in B_1(F, H),\ [\mu] = [\mu_0], [\nu] = [\nu_0]\}$$

where $[\mu_0], [\nu_0] \in T(F)$.

The *Kobayashi* (or *hyperbolic*) *pseudodistance* d_M on a complex manifold M is given as follows; for two points $p, q \in M$, choose points $p = p_0, p_1, \dots, p_{k-1}, p_k = q$ of M, points $a_1, \dots, a_k, b_1, \dots, b_k$ of Δ, and holomorphic mappings $f_1, \dots, f_k$ of Δ into M such that $f_i(a_i) = p_{i-1}$ and $f_i(b_i) = p_i$ for $i = 1, \dots, k$. Then

$$d_M(p, q) = \inf \sum_{i=1}^{k} \rho(a_i, b_i),$$

where ρ is the Poincaré distance on Δ and the infimum is taken with respect to all possible choices of finite points and holomorphic mappings as above. A complex manifold is called *hyperbolic* provided d_M is a distance. When $M = \Delta$, d_M coincides with ρ. For a finitely generated Fuchsian group of the first kind F, the Kobayashi pseudodistance $d_{T(F)}$ is a distance and coincides with the Teichmüller distance d. Let M and N be two complex manifolds and let $f\colon M \to N$ be a holomorphic mapping. Then from the definition of the Kobayashi pseudodistances,

$$d_M(p, q) \geq d_N(f(p), f(q)), \qquad p, q \in M.$$

3. Rigidity of holomorphic mappings into Teichmüller spaces

In this section, F denotes a finitely generated Fuchsian group of the first kind whose signature is (p, n) with $3p - 3 + n > 0$. The Teichmüller space $T(F)$ is identified with a bounded domain in C^{3p-3+n} via the Bers embedding. Let Δ be the unit disk $\{z \in C; |z| < 1\}$. We regard a holomorphic mapping $f\colon \Delta \to T(F)$ as a bounded holomorphic mapping $\Delta \to C^{3p-3+n}$. Hence the mapping f has radial limits at almost all points of $\partial\Delta = \{z \in C; |z| = 1\}$, and the limits belong to $\overline{T}(F)$. We begin with some lemmas.

Lemma 1 (Shiga [Sh1, Theorem 5]). *Let f be a holomorphic mapping of Δ into $T(F)$. Then the set*

$$E_{\text{cusp}} = \{e^{i\theta} \in \partial\Delta;\ f \textit{ has a radial limit } f_*(e^{i\theta}) \textit{ at } e^{i\theta}, \\ \textit{which corresponds to a cusp}\}$$

has linear measure 0.

Lemma 2 (Shiga [Sh2, Theorem 5]). *Let f be a holomorphic mapping of Δ into $\overline{T}(F)$. If $f(0)$ corresponds to a totally degenerate b-group, then the mapping f is a constant mapping, that is, $f \equiv f(0)$ on Δ.*

Remark. In fact, the statement of Shiga [Sh2] Theorem 5 is slightly different from that of the above lemma. But the same argument yields the above statement.

Lemma 2 yields another proof of the following lemma.

Lemma 3 (Bers [B3, Lemma 2]). *Let $\{\varphi_j\}$ and $\{\psi_j\}$ be sequences in $T(F)$ such that $d(\varphi_j, \psi_j) < M$ $(j = 1, 2, \ldots)$ for a universal constant $M < \infty$. If $\lim_{j\to\infty} \varphi_j = \varphi$ exists and φ corresponds to a totally degenerate group, then $\lim_{j\to\infty} \psi_j = \varphi$.*

Proof. For each j, there exists a holomorphic isometry $\Psi_j\colon \Delta \to T(F)$ such that $\Psi_j(0) = \varphi_j$ and $\Psi_j(\Delta) \ni \psi_j$. Then by the assumption,

$$\rho(0, \Psi_j^{-1}(\psi_j)) = d(\varphi_j, \psi_j) < M. \tag{3.1}$$

Here ρ is the Poincaré distance on Δ and d is the Teichmüller distance on $T(F)$. Hence the sequence $\{\Psi_j^{-1}(\psi_j)\}$ is contained in a compact set in Δ. Suppose that there exists a subsequence $\{\psi_{j_k}\}$ of $\{\psi_j\}$ such that $\lim_{k\to\infty} \psi_{j_k} \neq \varphi$. By taking a subsequence if necessary, we may assume that the sequence of mappings $\{\Psi_{j_k}\}$ converges to a holomorphic mapping $\Psi^0\colon \Delta \to \overline{T}(F)$ and that the sequence $\{\Psi_{j_k}^{-1}(\psi_{j_k})\}$ converges to a point $z_0 \in \Delta$. Then $\Psi^0(0) = \varphi$, which corresponds to a totally degenerate group, and $\Psi^0(z_0) \neq \varphi$. This contradicts Lemma 2.

Theorem 1. *Let* $f: \Delta \times \Delta \to T(F)$ *be a holomorphic mapping with the following property: there exists a point* $\zeta_0 \in \Delta$ *such that the set*

$$E_{\zeta_0} = \{e^{i\theta} \in \partial\Delta;\ f(\cdot\,, \zeta_0) \text{ has a radial limit } f_*(e^{i\theta}, \zeta_0) \text{ at } e^{i\theta} \text{ and that } f_*(e^{i\theta}, \zeta_0) \in \partial T(F)\}$$

has positive linear measure. Then f *depends only on the first variable* z, *i.e., for all* $\zeta, \zeta' \in \Delta$,

$$f(\cdot\,, \zeta) \equiv f(\cdot\,, \zeta') \quad \text{on } \Delta. \tag{3.2}$$

Proof. By Lemma 1, the measure of the set

$$E_{\zeta_0, \mathrm{cusp}} = \{e^{i\theta} \in E_{\zeta_0};\ f_*(e^{i\theta}, \zeta_0) \text{ corresponds to a cusp}\}$$

is 0. Set $\widehat{E}_{\zeta_0} = E_{\zeta_0} - E_{\zeta_0, \mathrm{cusp}}$. For each point $e^{i\theta} \in \widehat{E}_{\zeta_0}$ there exists a point $\phi_\theta \in \partial T(F)$ corresponding to a totally degenerate group such that

$$\lim_{r \to 1} f(re^{i\theta}, \zeta_0) = \phi_\theta. \tag{3.3}$$

Fix a point $\zeta \in \Delta$. Since the holomorphic mapping f does not increase the hyperbolic distances,

$$d(f(re^{i\theta}, \zeta_0), f(re^{i\theta}, \zeta)) \le d_{\Delta\times\Delta}((re^{i\theta}, \zeta_0), (re^{i\theta}, \zeta)) = d_\Delta(\zeta_0, \zeta). \tag{3.4}$$

For each increasing sequence $\{r_n\}$ with $0 < r_n < 1$ and $\lim_{n\to\infty} r_n = 1$, there exists a subsequence $\{r_{n_j}\}$ such that the limit $\lim_{n_j \to\infty} f(r_{n_j} e^{i\theta}, \zeta)$ exists. By (3.4) and Lemma 3, the limit is equal to ϕ_θ. Hence we have

$$\lim_{r \to 1} f(re^{i\theta}, \zeta) = \phi_\theta. \tag{3.5}$$

Since the mappings $f(\cdot\,, \zeta)$ and $f(\cdot\,, \zeta_0)$ are both bounded and have the same radial limit at each point of the positive measure set $\widehat{E}_{\zeta_o}$, $f(\cdot\,, \zeta) \equiv f(\cdot\,, \zeta_0)$ on Δ. Noting that ζ is an arbitrary point of Δ, the assertion of the theorem follows.

Remark 1. Lemma 2 yields another proof of Theorem 1, without using Kobayashi distance, which we omit here.

Remark 2. In connection with Theorem 1, there arises a natural question that the same statement as Theorem 1 is true for infinite-dimensional Teichmüller spaces. To this question, Professor C. McMullen suggested counterexamples. The details will be given elsewhere.

4. Corollaries to Theorem 1

In this section we derive corollaries from Theorem 1. Let F be a finitely generated Fuchsian group of the first kind. The following is immediate.

Corollary 1. *There exists no nontrivial family of proper holomorphic mappings of* Δ *into* $T(F)$ *with a complex analytic parameter.*

A holomorphic mapping $\widetilde{\Phi}: \Delta \to B_1(F, H)$ is projected to a holomorphic mapping $\Phi = \pi \circ \widetilde{\Phi}: \Delta \to T(F)$, here $\pi: B_1(F, H) \to T(F)$ is the canonical projection. The holomorphic mapping Φ has radial limits in $\overline{T}(F)$ almost everywhere in $\partial\Delta$. Let ω be a point of $\partial\Delta$ such that the radial limit

$\Phi_*(\omega) = \lim_{r\to 1} \Phi(r\omega)$ exists. Set $\widetilde{\Phi}(\zeta) = \mu_\zeta$ for $\zeta \in \Delta$. If $\Phi_*(\omega) \in \partial T(F)$, then $\lim_{r\to 1} \|\mu_{r\omega}\|_\infty = 1$. But of course the converse is not true in general. It is easy to construct a holomorphic mapping $\widetilde{\Phi}: \Delta \to B_1(F, H)$ such that $\lim_{r\to 1} \|\mu_{r\omega}\|_\infty = 1$ for all $\omega \in \partial\Delta$ and that the image $\Phi(\Delta)$ is relatively compact in $T(F)$. We claim that almost all radial limits belong to $T(F)$ provided each Beltrami differential $\mu_\zeta = \widetilde{\Phi}(\zeta)$, $\zeta \in \Delta$, vanishes on a fixed nontrivial set.

Corollary 2. *Let F be a finitely generated Fuchsian group of the first kind acting on H. Suppose that a holomorphic mapping $\widetilde{\Phi}: \Delta \to B_1(F, H)$ has the following property: there exists a F-invariant subset E of H with two-dimensional Lebesgue measure positive such that each Beltrami differential $\widetilde{\Phi}(\zeta) = \mu_\zeta$, $\zeta \in \Delta$, vanishes on E. Let $\Phi: \Delta \to T(F)$ be the projection of $\widetilde{\Phi}$ on $T(F)$. Then almost all radial limits of Φ on $\partial\Delta$ are contained in $T(F)$.*

Proof. In proving the corollary, we may assume that $\mu_0 = 0$. In fact it is well known (cf. [K2, §3.1]) that there exist a Fuchsian group F' and biholomorphic maps $\tilde{g}: B_1(F, H) \to B_1(F', H)$ and $g: T(F) \to T(F')$ such that $\tilde{g}(\mu_0) = 0$, $\pi' \circ \tilde{g} = g \circ \pi$, and $\pi' \circ \tilde{g} \circ \widetilde{\Phi} = g \circ \Phi$, where $\pi: B_1(F, H) \to T(F)$ and $\pi': B_1(F', H) \to T(F')$ are the natural projections. (Note that $\tilde{g}(B_1(F, E) = B_1(F, E')$ for some positive measure set E' since a quasi-conformal map is absolutely continuous.) To prove the corollary, it is sufficient to show that there exists a Beltrami differential ν in $B_1(F, E)$ such that the holomorphic mapping $\Psi: \Delta \times \Delta \to T(F)$ defined by

$$\Psi(z, \zeta) = [\mu_z + \zeta\nu],$$

depends on the second variable ζ nontrivially. Since the set E is a positive measure set, $B_1(F, E)$ is projected onto a neighborhood of the origin $[0]$ of $T(F)$ under the natural projection $B_1(F, E) \to T(F)$. Hence there exists a Beltrami differential $\nu \in B_1(F, E)$ such that $[\mu_0] \neq [\nu/2]$. Such a Beltrami differential has the desired property.

Corollary 3. *Let F be a finitely generated Fuchsian group of the first kind whose signature is (p, n) with $3p - 3 + n > 0$. Then the Teichmüller space $T(F)$ is not holomorphically equivalent to a direct product $M_1 \times M_2$, where M_i is a complex manifold of positive dimension, $i = 1, 2$.*

Proof. Assume that there exist complex manifolds M_1 and M_2 and a biholomorphic mapping $\Phi = (\Phi_1, \Phi_2)$ of $T(F)$ onto $M_1 \times M_2$, where Φ_i is a holomorphic mapping of $T(F)$ to M_i, $i = 1, 2$. Take a holomorphic proper mapping $h: \Delta \to T(F)$, and set $E = \{e^{i\theta} \in \partial\Delta;\, h$ has a radial limit $h_*(e^{i\theta})$ at $e^{i\theta}$ and that $h_*(e^{i\theta}) \in \partial T(F)\}$. For each $e^{i\theta} \in E$, set $c_\theta = \{re^{i\theta};\, 0 < r < 1\}$. Then either $\Phi_1 \circ h(c_\theta)$ is not relatively compact in M_1 or $\Phi_2 \circ h(c_\theta)$ is not relatively compact in M_2. Putting

$$E_i = \{e^{i\theta} \in E;\, \Phi_i \circ h(c_\theta) \text{ is not relatively compact in } M_i\}, \qquad i = 1, 2,$$

we have $E = E_1 \cup E_2$. Since $meas(E) > 0$, we may assume that $meas(E_1) > 0$, where $meas$ stands for the linear measure. For each point $p \in M_2$, define the holomorphic mapping $\Psi_p: \Delta \to T(F)$ by

$$\Psi_p(z) = \Phi^{-1}(\Phi_1 \circ h(z), p), \qquad z \in \Delta.$$

Then the set

$$E_1^p = \{e^{i\theta} \in E_1 ; \Psi_p \text{ has a radial limit } \Psi_{p*}(e^{i\theta})\}$$

has the same (hence positive) linear measure as E_1, and each $e^{i\theta} \in E_1^p$, $\Psi_{p*}(e^{i\theta}) \in \partial T(F)$. Note that the family of holomorphic mappings $\{\Psi_p\}_{p\in M_2}$ depends holomorphically on the parameter p. This contradicts Theorem 1.

Next we give a remark on holomorphic families over Riemann surfaces of finite type. We begin with a brief description. See for example Imayoshi-Shiga [IS], which contains an analytic proof of Parsin-Arakerov's theorem, for more details and references. Let F_1 and F_2 be finitely generated torsion free Fuchsian groups of the first kind acting on Δ, and let (p, n) be the signature of F_2. A holomorphic mapping Φ of Δ into $T(F_2)$ is called a holomorphic family of type (p, n) over the Riemann surface Δ/F_1 if for each $g \in F_1$ there exists a holomorphic homeomorphism $\chi(g)$ of $T(F_2)$ such that $\Phi\circ g = \chi(g)\circ\Phi$. The correspondence $g \to \chi(g)$ is a homomorphism. Two holomorphic families Φ_1 and Φ_2 are called equivalent if there exists a holomorphic homeomorphism γ of $T(F_2)$ such that $\Phi_1 = \gamma\circ\Phi_2$. Noguchi [N] investigated the set of holomorphic mappings of certain kinds of complex spaces and showed that the set of holomorphic families over Δ/F_1 is a Zariski open subset of a compact complex space, hence consists of finitely many components. If a holomorphic family Φ is not a constant map, from the argument in [IS] (from p. 212 line 32 to p. 213 line 18), radial limits of Φ at almost everywhere in $\partial\Delta$ belong to $\partial T(F)$. Hence from Theorem 1 it follows that the component containing Φ consists of one point. Thus we get another proof of Parshin-Arakelov's finiteness theorem of holomorphic families;

Corollary 4. *For given finitely generated torsion free Fuchsian group of the first kind F_1 and a pair of nonnegative integers (p, n) with $2p-2+n > 0$, there exists only finitely many nonequivalent and locally nontrivial holomorphic families of type (p, n) over Δ/F_1.*

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