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REFINEMENTS OF KY FAN'S INEQUALITY

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ABSTRACT. We prove the inequalities

$$A'_n/G'_n \le (1-G'_n)/(1-A'_n) \le A_n/G_n$$

and

$$A'_n/G'_n \le (1-G_n)/(1-A_n) \le A_n/G_n,$$

where A_n and G_n (respectively, A'_n and G'_n) denote the unweighted arithmetic and geometric means of $x_1, \dots, x_n$ (respectively, $1-x_1, \dots, 1-x_n$) with $x_i \in (0, \frac{1}{2}]$ $(i = 1, \dots, n;\ n \ge 2)$. Further we show that the ratios $(1-G'_n)/(1-A'_n)$ and $(1-G_n)/(1-A_n)$ can be compared if and only if $n = 2$.

1. INTRODUCTION

In 1961 the following remarkable inequality, due to Ky Fan, was published for the first time in the well-known book *Inequalities* by Beckenbach and Bellman [3, p. 5]:

If A_n and G_n (respectively, A'_n and G'_n) denote the unweighted arithmetic and geometric means of the real numbers $x_1, \dots, x_n$ (respectively, $1-x_1, \dots, 1-x_n$), i.e.,

$$A_n = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad G_n = \prod_{i=1}^{n} x_i^{1/n}$$

(respectively, $A'_n = \frac{1}{n}\sum_{i=1}^{n}(1-x_i)$ and $G'_n = \prod_{i=1}^{n}(1-x_i)^{1/n}$), then we have for all $x_i \in (0, \frac{1}{2}]$ $(i = 1, \dots, n;\ n \ge 2)$,

$$G_n/G'_n \le A_n/A'_n. \tag{1.1}$$

Equality holds in (1.1) if and only if $x_1 = \cdots = x_n$.

Inequality (1.1) has evoked the interest of several mathematicians and many papers have been published providing new proofs, noteworthy extensions, and sharpenings as well as intriguing counterparts and variants; see [2] and the references therein.

Among the different refinements of Fan's inequality we could not find one presenting a sharpening of the equivalent inequality

$$A'_n/G'_n \le A_n/G_n. \tag{1.2}$$

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The aim of this paper is to prove two refinements of inequality (1.2). In §2 we establish that the ratios $(1-G'_n)/(1-A'_n)$ and $(1-G_n)/(1-A_n)$ both separate the left-hand side and the right-hand side of (1.2). It is natural to ask whether all four quotients can be included in a chain of three inequalities. This is indeed possible if $n = 2$; but if $n > 2$, then the expressions $(1-G'_n)/(1-A'_n)$ and $(1-G_n)/(1-A_n)$ cannot be compared. These results will be proved in §3.

2. Two refinements

In the proof of Theorem 1 the following additive analogue of inequality (1.1) plays a central role.

If $x_i \in (0, \frac{1}{2}]$ $(i = 1, \dots, n;\ n \geq 2)$ then

$$G_n - G'_n \leq A_n - A'_n, \tag{2.1}$$

with equality holding if and only if $x_1 = \dots = x_n$.

A proof for this proposition can be found in [1].

Theorem 1. *If* $x_i \in (0, \frac{1}{2}]$ $(i = 1, \dots, n;\ n \geq 2)$ *then*

$$A'_n/G'_n \leq (1-G'_n)/(1-A'_n) \leq A_n/G_n. \tag{2.2}$$

Equality is valid if and only if $x_1 = \dots = x_n$.

Proof. The function $f(x) = x(1-x)$ is strictly decreasing on $[\frac{1}{2}, \infty)$. Because of $\frac{1}{2} \leq G'_n \leq A'_n < 1$, we obtain $f(A'_n) \leq f(G'_n)$ with equality holding if and only if all the x_i's are equal. This establishes the left-hand side of (2.2).

Since $A_n + A'_n = 1$, we obtain from (2.1) that

$$G_n(1-G'_n) \leq G_n(2A_n - G_n) \leq A_n^2, \tag{2.3}$$

which yields the second inequality of (2.2). If $G_n(1-G'_n) = A_n^2$ then we conclude from the right-hand inequality of (2.3): $A_n = G_n$; hence $x_1 = \dots = x_n$. □

Remark. From double-inequality (2.3) we get the following sharpening of the right-hand side of (2.2):

$$(1-G'_n)/(1-A'_n) \leq 2 - G_n/A_n \leq A_n/G_n. \tag{2.4}$$

Equality is valid if and only if all the x_i's are equal. This is obvious for the second inequality of (2.4), and since equality holds in (2.1) only if $x_1 = \dots = x_n$, the same is true for the first inequality of (2.4).

Theorem 2. *If* $x_i \in (0, \frac{1}{2}]$ $(i = 1, \dots, n;\ n \geq 2)$ *then*

$$A'_n/G'_n \leq (1-G_n)/(1-A_n) \leq A_n/G_n, \tag{2.5}$$

with equality holding if and only if $x_1 = \dots = x_n$.

Proof. The validity of the second inequality follows immediately from $0 < G_n \leq A_n \leq \frac{1}{2}$ and the fact that $f(x) = x(1-x)$ is strictly increasing on $(0, \frac{1}{2}]$.

To establish the left-hand inequality of (2.5) we define

$$g\colon [0, \tfrac{1}{2}]^n \to \mathbb{R},$$

$$g(x_1, \dots, x_n) = \left(1 - \prod_{i=1}^n x_i^{1/n}\right) \prod_{i=1}^n (1-x_i)^{1/n} - \left(1 - \frac{1}{n}\sum_{i=1}^n x_i\right)^2.$$

Let $\underline{a} = (a_1, \ldots, a_n) \in [0, \frac{1}{2}]^n$ be the absolute minimum of g. We prove $a_1 = \cdots = a_n$, which implies

$$g(x_1, \ldots, x_n) \geq g(a_1, \ldots, a_1) = 0 \quad \text{for all } (x_1, \ldots, x_n) \in \left[0, \tfrac{1}{2}\right]^n$$

with equality holding if and only if $x_1 = \cdots = x_n$.

If $\underline{a}$ is an interior point of $[0, \frac{1}{2}]^n$, then we obtain

$$\nabla g(a_1, \ldots, a_n) = 0$$

such that $a_1, \ldots, a_n$ solve the equation

$$P(x) = -G_n G'_n(1-x) - (1-G_n)G'_n x + 2(1-A_n)x(1-x) = 0.$$

Since P is a polynomial of degree 2, we conclude from

$$P(0) < 0 \quad \text{and} \quad 2P\left(\tfrac{1}{2}\right) = 1 - G'_n - A_n \geq 1 - A'_n - A_n = 0$$

that P has at most one zero on $(0, \frac{1}{2})$; hence $a_1 = \cdots = a_n$.

Next we assume that $\underline{a}$ is a boundary point of $[0, \frac{1}{2}]^n$. We consider two cases.

Case 1. No component of $\underline{a}$ is equal to 0. Then l (≥ 1) components of $\underline{a}$ are equal to $\frac{1}{2}$. Without loss of generality, we may suppose

$$a_{k+1} = \cdots = a_n = \tfrac{1}{2}, \qquad 1 \leq n - k = l \leq n - 1.$$

We define

$$h\colon \left[0, \tfrac{1}{2}\right]^k \to \mathbb{R},$$

$$\begin{aligned} h(x_1, \ldots, x_k) &= g\left(x_1, \ldots, x_k, \tfrac{1}{2}, \ldots, \tfrac{1}{2}\right) \\ &= \tfrac{1}{2}[1 - \tfrac{1}{2}(2G_k)^{k/n}](2G'_k)^{k/n} - \left[\tfrac{1}{2} + k\left(\tfrac{1}{2} - A_k\right)/n\right]^2. \end{aligned}$$

Because of

$$h(x_1, \ldots, x_k) \geq h(a_1, \ldots, a_k) \quad \text{for all } (x_1, \ldots, x_k) \in \left[\dot{0}, \tfrac{1}{2}\right]^k, \tag{2.6}$$

we conclude that h attains its absolute minimum at $\tilde{\underline{a}} = (a_1, \ldots, a_k)$. Since $0 < a_i < \frac{1}{2}$ $(i = 1, \ldots, k)$, we obtain $\nabla h(a_1, \ldots, a_k) = 0$, which implies that $a_1, \ldots, a_k$ solve the equation

$$\begin{aligned} Q(x) = {} & \tfrac{1}{4}(4G_k G'_k)^{k/n}(2x-1) - \tfrac{1}{2}(2G'_k)^{k/n}x \\ & + (-2kA_k/n + 1 + k/n)\,x(1-x) = 0. \end{aligned}$$

We have $Q(0) < 0$ and

$$4Q\left(\tfrac{1}{2}\right) = -(2G'_k)^\alpha - 2A_k\alpha + 1 + \alpha \tag{2.7}$$

with $\alpha = k/n \in (0, 1)$. If we designate the right-hand side of (2.7) by $\widetilde{Q}(\alpha)$, then $\widetilde{Q}$ is strictly concave on $[0, 1]$ and, since $\widetilde{Q}(0) = 0$ and

$$\widetilde{Q}(1) = 2(1 - A_k - G'_k) \geq 2(1 - A_k - A'_k) = 0,$$

we conclude

$$Q\left(\tfrac{1}{2}\right) = \tfrac{1}{4}\widetilde{Q}(k/n) > 0.$$

Thus, Q has precisely one root on $(0, \frac{1}{2})$, which leads to $a_1 = \cdots = a_k$. Now we prove that the function

$$\tilde{h}(x) = h(x, \ldots, x)$$

is strictly decreasing on $[0, \frac{1}{2}]$. This implies

$$h(a_1, \dots, a_k) = \widetilde{h}(a_1) > \widetilde{h}\left(\tfrac{1}{2}\right) = h\left(\tfrac{1}{2}, \dots, \tfrac{1}{2}\right),$$

which contradicts inequality (2.6). Differentiation of $\widetilde{h}$ yields, for $x \in (0, \frac{1}{2})$,

$$\text{(2.8)} \quad \frac{1}{\alpha}\widetilde{h}'(x) = \frac{1}{4}\left(\frac{1}{1-x} - \frac{1}{x}\right)[4x(1-x)]^{\alpha} - \frac{1}{2(1-x)}[2(1-x)]^{\alpha} + 1 + \alpha - 2\alpha x$$

with $\alpha = k/n \in (0, 1)$. We denote the right-hand side of (2.8) by $p(\alpha)$. Differentiation of p leads to

$$\begin{aligned} p''(\alpha) = {} & (2x-1)[4x(1-x)]^{\alpha-1}[\log(4x(1-x))]^2 \\ & - [2(1-x)]^{\alpha-1}[\log(2(1-x))]^2 < 0. \end{aligned}$$

Hence we obtain, for $\alpha \in (0, 1)$:

$$\text{(2.9)} \qquad p'(\alpha) > p'(1) = (2x-1)\log(4x(1-x)) - \log(2(1-x)) + 1 - 2x.$$

We designate the right-hand side of (2.9) by $q(x)$. Because of $q''(x) > 0$ for $x \in (0, \frac{1}{2})$ and $q(\frac{1}{2}) = q'(\frac{1}{2}) = 0$, we conclude $p'(1) > 0$. Therefore $p(\alpha) < p(1) = 0$ for $\alpha \in (0, 1)$, which proves that $\widetilde{h}$ is strictly decreasing on $[0, \frac{1}{2}]$.

Case 2. l (≥ 1) components of $\underline{a}$ are equal to 0. We assume

$$a_{k+1} = \cdots = a_n = 0, \qquad 1 \leq n - k = l \leq n - 1,$$

and define

$$\varphi\colon \left[0, \tfrac{1}{2}\right]^k \to \mathbb{R},$$

$$\varphi(x_1, \dots, x_k) = g(x_1, \dots, x_k, 0, \dots, 0) = \prod_{i=1}^{k}(1 - x_i)^{1/n} - \left(1 - \frac{1}{n}\sum_{i=1}^{k} x_i\right)^2.$$

We have, for $j = 1, \dots, k$,

$$\frac{n}{2}\varphi_{x_j}(x_1, \dots, x_k) = -\frac{1}{2(1-x_j)}(G_k')^{\alpha} + 1 - \alpha A_k \geq -(G_k')^{\alpha} + 1 - \alpha A_k$$

with $\alpha = k/n \in (0, 1)$. Since the function

$$\psi(\alpha) = -(G_k')^{\alpha} + 1 - \alpha A_k$$

is strictly concave on $[0, 1]$ and because of

$$\psi(0) = 0 \quad \text{and} \quad \psi(1) = -G_k' + 1 - A_k = -G_k' + A_k' \geq 0,$$

we obtain

$$\psi(\alpha) > 0 \quad \text{for } \alpha \in (0, 1).$$

Hence we have

$$\varphi(x_1, \dots, x_k) \geq \varphi(0, \dots, 0) = 0 \quad \text{for all } (x_1, \dots, x_k) \in \left[0, \tfrac{1}{2}\right]^k.$$

Since φ attains its absolute minimum at $\underline{\tilde{a}} = (a_1, \dots, a_k)$, we conclude $a_1 = \cdots = a_k = 0$. This completes the proof of Theorem 2. □

3. The case $n = 2$

In this section we prove that the ratios $(1-G_n')/(1-A_n')$ and $(1-G_n)/(1-A_n)$ can be compared if and only if $n = 2$.

Theorem 3. *If* $x_1, x_2 \in (0, \frac{1}{2}]$ *then*

$$A_2'/G_2' \leq (1 - G_2')/(1 - A_2') \leq (1 - G_2)/(1 - A_2) \leq A_2/G_2, \tag{3.1}$$

with equality holding if and only if $x_1 = x_2$.

Proof. It remains to establish the second inequality of (3.1). We define

$$f\colon \left[0, \tfrac{1}{2}\right]^2 \to \mathbb{R},$$

$$f(x, y) = (1 - \sqrt{xy})\frac{x+y}{2} - \left(1 - \frac{x+y}{2}\right)\left(1 - \sqrt{(1-x)(1-y)}\right),$$

and denote the absolute minimum of f by $\underline{a} = (a_1, a_2)$. We prove $a_1 = a_2$. If $\underline{a}$ is an interior point of $[0, \frac{1}{2}]^2$ then we have

$$\nabla f(a_1, a_2) = 0,$$

which leads to

$$-\sqrt{a_2/a_1}A_2 + 2 - G_2 - G_2' - \sqrt{(1-a_2)/(1-a_1)}A_2' = 0$$

and

$$-\sqrt{a_1/a_2}A_2 + 2 - G_2 - G_2' - \sqrt{(1-a_1)/(1-a_2)}A_2' = 0.$$

From these equations we obtain

$$(A_2/G_2 - A_2'/G_2')(a_1 - a_2) = 0.$$

Suppose $a_1 \neq a_2$. Then we get $A_2/G_2 = A_2'/G_2'$, and from Fan's theorem we conclude $a_1 = a_2$.

Next we assume that $\underline{a}$ is a boundary point of $[0, \frac{1}{2}]^2$. We distinguish two cases.

Case 1. One component of $\underline{a}$ is equal to 0. If $a_1 = 0$ and $a_2 = z \in (0, \frac{1}{2}]$, then we have

$$F(z) = f(0, z) = z - 1 + \sqrt{1-z}\left(1 - \frac{z}{2}\right)$$

and

$$2\sqrt{1-z}F'(z) = \frac{3z}{2} - 2 + 2\sqrt{1-z}. \tag{3.2}$$

Since the right-hand side of (3.2) is increasing on $[0, \frac{1}{2}]$ we get, for $z \in (0, \frac{1}{2}]$, $F'(z) > 0$ and $F(z) > F(0) = 0$.

Case 2. Both components of $\underline{a}$ are different from 0. Let $a_1 = \frac{1}{2}$ and $a_2 = z \in (0, \frac{1}{2}]$. Then we have

$$G(z) = 4f\left(\frac{1}{2}, z\right) = \left(1 - \sqrt{\frac{z}{2}}\right)(1 + 2z) - (3 - 2z)\left(1 - \sqrt{\frac{1-z}{2}}\right).$$

A simple calculation reveals

$$2\sqrt{2}(1-z)^{3/2}G''(z) = \left(\frac{1-z}{z}\right)^{3/2}\left(\frac{1}{2}-3z\right) + \frac{5}{2} - 3z > 0 \quad \text{for } z \in \left(0, \frac{1}{2}\right]$$

and

$$G\left(\tfrac{1}{2}\right) = G'\left(\tfrac{1}{2}\right) = 0,$$

which implies

$$G(z) \geq 0 \quad \text{for } z \in \left(0, \tfrac{1}{2}\right],$$

with equality holding if and only if $z = \frac{1}{2}$. □

Now we show that for every $n \geq 3$ the ratios $(1-G_n')/(1-A_n')$ and $(1-G_n)/(1-A_n)$ cannot be compared. If we set $x_1 = \cdots = x_{n-1} = 0$ and $x_n = \frac{1}{2}$, then

$$(1-G_n')/(1-A_n') > (1-G_n)/(1-A_n)$$

is equivalent to

$$((2n-2)/(2n-1))^n > \tfrac{1}{2}.$$

Since $\alpha_n = ((2n-2)/(2n-1))^n$ is strictly increasing, we obtain, for $n \geq 3$,

$$\alpha_n \geq \alpha_3 = \tfrac{64}{125} > \tfrac{1}{2}.$$

Next we put $x_1 = 0$ and $x_2 = \cdots = x_n = \frac{1}{2}$. Then

$$(3.3) \qquad (1-G_n')/(1-A_n') < (1-G_n)/(1-A_n)$$

and $\frac{1}{2} < ((n+1)/4)^n$ are equivalent. The sequence $\beta_n = ((n+1)/4)^n$ is strictly increasing, which implies, for $n \geq 2$,

$$\beta_n \geq \beta_2 = \tfrac{9}{16} > \tfrac{1}{2}.$$

The fact that inequality (3.3) is valid for $n = 2$, but not for $n > 2$, is rather unusual since "most classical inequalities follow inductively from the two dimensional theorem" [4, p. 206]. Another example of this kind—also in connection with Fan's inequality—was given in 1974 by F. Chan, D. Goldberg, and S. Gonek [4]. They proved that the function

$$f(r; x_1, \ldots, x_n) = \begin{cases} \left(\sum_{i=1}^{n} x_i^r \Big/ \sum_{i=1}^{n} (1-x_i)^r\right)^{1/r}, & r \neq 0, \\ \prod_{i=1}^{n} [x_i/(1-x_i)]^{1/n}, & r = 0, \end{cases}$$

satisfies the inequality

$$(3.4) \qquad f(r; x_1, x_2) < f(s; x_1, x_2)$$

for all real r and s with $r < s$ and for all nonnegative x_1, x_2 with $x_1 + x_2 < 1$ and $x_1 \neq x_2$. This result, in particular, leads to

$$G_2/G_2' = f(0; x_1, x_2) < f(r; x_1, x_2) < f(1; x_1, x_2) = A_2/A_2'$$

for all $r \in (0, 1)$.

Furthermore, Chan, Goldberg, and Gonek investigated the question whether inequality (3.4) also holds for more than two variables. Presenting interesting counterexamples, they established that the implication

$$r < s \Rightarrow f(r; x_1, \dots, x_n) \le f(s; x_1, \dots, x_n), \quad x_i \in (0, \tfrac{1}{2}] \ (i = 1, \dots, n), \tag{3.5}$$

is in general not true if $n > 2$. However, in a recently published paper [2] it was proved that (3.5) is valid for all $n \ge 2$ if $0 \le r < s \le 1$, which yields a refinement of Fan's inequality written in the form (1.1).

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