PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 117, Number 1, January 1993

L^p FUNCTION DECOMPOSITION ON C^∞ TOTALLY REAL SUBMANIFOLDS OF $\mathbb{C}^n$

WILLIAM S. CALBECK

(Communicated by Clifford J. Earle, Jr.)

ABSTRACT. For $1 < p < \infty$ we show that L^p functions defined on a C^∞ totally real submanifold of $\mathbb{C}^n$ can be locally decomposed into the sum of boundary values of holomorphic functions in wedges such that the boundary values are in L^p.

The general case of a C^∞ totally real submanifold is reduced to the flat case of $\mathbb{R}^n$ in $\mathbb{C}^n$ by an almost analytic change of variables. L^p results in the flat case are then obtained using Fourier multipliers. In transporting these results back to the manifold we lose analyticity, so it is necessary to solve a $\overline{\partial}$ problem in an appropriate domain. This gives holomorphy in the wedges but produces a C^∞ error on the edge. This C^∞ function is then holomorphically decomposed using the FBI transform with a careful analysis to check that the functions are C^∞ up to the edge and do not destroy the L^p behavior.

0. INTRODUCTION

A distribution defined on a totally real C^∞ submanifold of $\mathbb{C}^n$ can be decomposed locally into the sum of boundary values of functions that are holomorphic in wedges. These holomorphic functions have polynomial growth near the edge and so the boundary values are understood in the sense of distributions. This result is found in the paper of Baouendi, Chang, and Treves [1]. In this paper we intend to show that *if a function defined on a totally real C^∞ submanifold of $\mathbb{C}^n$ is in an L^p class for $1 < p < \infty$, then it can be decomposed so that the holomorphic functions have L^p boundary values* (Theorem 2). We first show by careful estimates using the FBI transform that a C^∞ function on the submanifold can be decomposed as in [1] so that the holomorphic functions are C^∞ up to the edge. The problem of L^p decomposition is then reduced to the problem of C^∞ decomposition. This paper is mainly concerned with $n > 1$, however the method can be applied to the case $n = 1$, which corresponds to a curve in $\mathbb{C}$.

1. FLAT CASE

The straight case is $\mathbb{R}^n$ as a totally real submanifold of $\mathbb{C}^n$ for which the L^p decomposition is already known. For the convenience of the reader we sketch a proof of this fact and include some definitions that will be used later.

Received by the editors January 28, 1991 and, in revised form, May 24, 1991.
1991 *Mathematics Subject Classification*. Primary 32F25; Secondary 32D15.

Definitions. A subset $\Gamma \subseteq \mathbb{R}^n \backslash \{0\}$ is a *cone* if $x \in \Gamma$ and $\lambda > 0$ implies $\lambda x \in \Gamma$. The *polar cone* of Γ is $\Gamma^o = \{y \in \mathbb{R}^n | x \cdot y > 0, \forall x \in \Gamma\}$. If Γ_1 and Γ_2 are two cones we say $\Gamma_1 \Subset \Gamma_2$ if the closure of Γ_1 is contained in $\Gamma_2 \cup \{0\}$.

Proposition 1. *Let* $\Gamma_0, \Gamma_1, \dots, \Gamma_\nu, C_0, C_1, \dots, C_\nu$ *be a collection of nonempty open cones such that* $C_j \Subset \Gamma_j^o$ *for* $j = 0, 1, \dots, \nu$ *and the* C_j *cover* $\mathbb{R}^n \backslash \{0\}$. *If* $1 < p < \infty$ *then every* $f \in L^p(\mathbb{R}^n)$ *can be written as* $f = \sum_{j=0}^{\nu} bv\, f_j$ *where* $bv\, f_j \in L^p(\mathbb{R}^n)$ *is the boundary value of a holomorphic function* f_j *defined in the* wedge $\mathbb{R}^n + i\Gamma_j$, *for* $j = 0, 1, \dots, \nu$.

Proof. Since the C_j cover $\mathbb{R}^n \backslash \{0\}$, there exists a partition of unity $1 = \sum_{j=0}^{\nu} \chi_j(\xi)$ where $\chi_j(\xi) \in C^\infty(\mathbb{R}^n \backslash \{0\})$, $\operatorname{supp} \chi_j(\xi) \subseteq C_j \cup \{0\}$, and for each j, $\chi_j(\xi)$ is positively homogeneous of degree 0. Let $f \in L^p(\mathbb{R}^n)$. Then

$$f(x) = (\hat{f}(\xi)^\vee(x) = \sum_{j=0}^{\nu} (\chi_j(\xi)\hat{f}(\xi))^\vee(x) = \sum_{j=0}^{\nu} bv\, f_j(x).$$

For $j = 0, 1, \dots, \nu$, $bv\, f_j(x)$ can be extended holomorphically to the wedge $\mathbb{R}^n + i\Gamma_j$ by the formula

$$f_j(x+iy) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \chi_j(\xi) e^{i(x+iy)\cdot\xi} \hat{f}(\xi)\, d\xi.$$

The integral will converge absolutely for $y \in \Gamma_j$ and the map taking f to $bv\, f_j$ for each j is a homogeneous Fourier multiplier operator that maps $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ (see for instance Stein [3, p. 96]).

2. Curve case

We now generalize Proposition 1 to the case of a C^∞ totally real submanifold of $\mathbb{C}^n$. It is first necessary to prove the splitting of C^∞ functions. This will be Theorem 1, which we state now but prove in the next section. Theorem 2, which is the main result of this paper concerning L^p splitting, is proven here and makes use of Proposition 1 and Theorem 1.

Definitions. After a local holomorphic change of variables a C^∞ *totally real submanifold* M of $\mathbb{C}^n$ is locally the image of a C^∞ map $Z = x + i\phi(x)$ from $\mathbb{R}^n$ into $\mathbb{C}^n$ such that ϕ has compact support, $\phi(0) = 0$, $\phi'(0) = 0$, and $\phi''(0) = 0$. From now on M is a fixed C^∞ totally real submanifold with local defining function Z as described. Let $Z_\sharp$ be an *almost analytic* extension of Z to all of $\mathbb{C}^n$. Such an extension is a C^∞ mapping $Z_\sharp \colon \mathbb{C}^n \to \mathbb{C}^n$ such that $Z_\sharp(x) = Z(x)$ for every $x \in \mathbb{R}^n$ and $(\partial/\partial \bar{z}_j) Z_\sharp^k$ vanishes to infinite order at $\operatorname{Im} z = 0$ for all $j, k = 1, \dots, n$. We note here for later use that all derivatives of $(\partial/\partial \bar{z}_j) Z_\sharp^k$ will also vanish to infinite order at $\operatorname{Im} z = 0$ since $Z_\sharp^k$ is C^∞ there. If Γ is a cone in $\mathbb{R}^n \backslash \{0\}$ and Θ is a neighborhood of 0 in $\mathbb{C}^n$ we define the *wedge* $\mathrm{W}(\Gamma, \Theta) = \{Z_\sharp(x+iy) : x+iy \in (\mathbb{R}^n + i\Gamma) \cap \Theta\}$ having *edge* $E(\Theta) = Z_\sharp(\mathbb{R}^n \cap \Theta) \subset M$.

Theorem 1 (C^∞ splitting). *Let* $\Gamma_0, \Gamma_1, \dots, \Gamma_\nu, C_0, C_1, \dots, C_\nu$ *be a collection of nonempty open cones such that* $C_j \Subset \Gamma_j^0$ *for* $j = 0, 1, \dots, \nu$, *and the* C_j *cover* $\mathbb{R}^n \backslash \{0\}$. *Let* Θ *be an open neighborhood of* 0 *in* C^n. *Then there exists an open neighborhood* Θ' *of* 0 *such that* $\Theta' \subset \Theta$ *and every* $u \in C^\infty(E(\Theta))$

can be written on $E(\Theta')$ *as* $u = \sum_{j=0}^{\nu} bvu_j$ *where for* $j = 0, 1, \ldots, \nu$, $bvu_j \in C^\infty(E(\Theta'))$ *is the boundary value of a holomorphic function* u_j *in the wedge* $W(\Gamma_j, \Theta')$ *and is* C^∞ *up to the edge* $E(\Theta')$.

Proof. To be given in the next section.

Theorem 2 (L^p splitting). *Let* $\Gamma_0, \Gamma_1, \ldots, \Gamma_\nu, C_0, C_1, \ldots, C_\nu$ *be a collection of nonempty open cones such that* $C_j \Subset \Gamma_j^0$ *for* $j = 0, 1, \ldots, \nu$, *and the* C_j *cover* $\mathbb{R}^n \backslash \{0\}$. *Let* $1 < p < \infty$ *and let* Θ *be an open neighborhood of* 0 *in* $\mathbb{C}^n$. *Then there exists an open neighborhood* Θ' *of* 0 *such that* $\Theta' \subset \Theta$ *and every* $f \in L^p(E(\Theta))$ *can be written on* $E(\Theta')$ *as* $f = \sum_{j=0}^{\nu} bvf_j$ *where for* $j = 0, 1, \ldots, \nu$, $bvf_j \in L^p(E(\Theta'))$ *is the boundary value of a holomorphic function* f_j *in the wedge* $W(\Gamma_j, \Theta')$.

Proof. Step 1. Almost analytic splitting. Without loss of generality, assume Θ is so small that $Z_\sharp$ restricted to Θ is a diffeomorphism on to $Z_\sharp(\Theta)$. For $j = 0, 1, \ldots, \nu$, we can enlarge Γ_j to $\widetilde{\Gamma}_j$ such that $\widetilde{\Gamma}_j$ is strictly convex, $\Gamma_j \Subset \widetilde{\Gamma}_j$ and $C_j \Subset \widetilde{\Gamma}_j^0$. Since $C_j \Subset \Gamma_j^0$, there exists a constant c_j such that $\forall x \in C_j$ and $\forall y \in \Gamma_j$, $x \cdot y > c_j|x|\,|y|$, so we can take $\widetilde{\Gamma}_j = \{y \in \mathbb{R}^n;\ x \cdot y > (\frac{1}{2})c_j|x|\,|y|, \forall x \in C_j\}$. Let $f \in L^p(E(\Theta))$. Then $g(x) = f(Z(x)) \in L^p(\mathbb{R}^n \cap \Theta)$. Extend g by 0 to $\mathbb{R}^n$; then $g \in L^p(\mathbb{R}^n)$, so according to Proposition 1 we can write g as $g = \sum_{j=0}^{\nu} bvg_j$ where for $j = 0, 1, \ldots, \nu$, $bvg_j \in L^p(\mathbb{R}^n)$ is the boundary value of a holomorphic function g_j in the wedge $\mathbb{R}^n + i\widetilde{\Gamma}_j$. For $j = 0, \ldots, \nu$, define $f_j(w) = g_j(Z_\sharp^{-1}(w))$ for $w \in W(\widetilde{\Gamma}_j, \Theta)$.

Since each g_j is holomorphic in the wedge $\mathbb{R}^n + i\widetilde{\Gamma}_j$ and $Z_\sharp$ is almost analytic, f_j is almost analytic in the wedge $W(\widetilde{\Gamma}_j, \Theta)$, meaning that $(\partial/\partial \bar{z}_k)f_j$ vanishes to infinite order at the edge $E(\Theta)$ for all $k = 1, \ldots, n$. Therefore every $f \in L^p(E(\Theta))$ can be written on $E(\Theta)$ as $f = \sum_{j=0}^{\nu} bvf_j$ where for $j = 0, 1, \ldots, \nu$, $bvf_j \in L^p(E(\Theta))$ is the boundary value of a function f_j that is almost analytic in the wedge $W(\widetilde{\Gamma}_j, \Theta)$.

Here we make a note of the polynomial growth of the derivatives of g_j and therefore of f_j at the edge. Now $g \in L^p(\mathbb{R}^n)$ and has compact support so $\hat{g} \in L^\infty(\mathbb{R}^n)$. Let α be a multi-index from $\mathbb{N}^n$ then

$$\left(\frac{\partial}{\partial z}\right)^\alpha g_j(x+iy) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \chi_j(\xi)(i\xi)^\alpha e^{i(x+iy)\cdot\xi}\hat{g}(\xi)\,d\xi.$$

Therefore

$$\left|\left(\frac{\partial}{\partial z}\right)^\alpha g_j(x+iy)\right| \le \frac{C_\alpha}{|y|^{|\alpha|+n}}$$

where

$$C_\alpha = \frac{\|\hat{g}\|_\infty}{(2\pi)^n}\int_{\mathbb{R}^n} |\xi|^{|\alpha|} e^{-c_j|\xi|}\,d\xi.$$

Step 2. $\bar\partial$ correction, which leads to a holomorphic splitting modulo a C^∞ function on the edge. We now construct a domain Ω contained in the wedge $W(\widetilde{\Gamma}_0, \Theta)$ on which to solve $\bar\partial u_0 = \bar\partial f_0$. The domain Ω will be the intersection of finitely many strictly pseudoconvex domains and for such domains Dufresnoy [2] prove that if $\bar\partial f_0$ is C^∞ up to the boundary, then the solution u_0 can be chosen to be C^∞ up to the boundary.

Since $\Gamma_0 \Subset \widetilde{\Gamma}_0$ and $\widetilde{\Gamma}_0$ is strictly convex, there exists a finite number of vectors $v_1, \dots, v_m$ and corresponding half spaces $H_1, \dots, H_m$ of $\mathbb{R}^n$, where $H_j = \{x \in \mathbb{R}^n ; x \cdot v_j > 0\}$, such that

$$\Gamma_0 \Subset \bigcap_{j=1}^{m} H_j \Subset \widetilde{\Gamma}_0 .$$

For each $j = 1, \dots, m$ let $\rho_j(z) = \rho_j(x+iy) = |y|^2 - y \cdot v_j$, which is a strictly plurisubharmonic function in $\mathbb{C}^n$. The region $\{\rho_j < 0\}$ is a tube domain over $i\mathbb{R}^n$ whose base projection is a ball contained in H_j and tangent to the boundary of H_j, the set $\{x \cdot v_j = 0\}$, at 0. Near 0, $\mathbb{R}^n + iH_j \approx \{\rho_j < 0\}$, so there exists a neighborhood $U \subset \Theta$ of 0 in $\mathbb{C}^n$ such that

$$(\mathbb{R}^n + i\Gamma_0) \cap U \Subset \bigcap_{j=1}^{m} \{\rho_j < 0\} \cap U \Subset (\mathbb{R}^n + i\widetilde{\Gamma}_0) \cap U .$$

Because $Z_\sharp$ is almost analytic at 0 and each ρ_j is strictly plurisubharmonic, there exists a neighborhood $V \subset U$ of 0 in $\mathbb{C}^n$ such that for $j = 1, \dots, m$, $\rho_j(Z_\sharp^{-1}(w))$ is strictly plurisubharmonic on $Z_\sharp(V)$. Let $r > 0$ such that the ball $B(0, r) \subset Z_\sharp(V)$ and set

$$\Omega = \{w \in B(0, r) ; \rho_j(Z_\sharp^{-1}(w)) < 0 \text{ for } j = 1, \dots, m\} .$$

Then Ω is the intersection of $m+1$ strictly pseudoconvex domains and

$$Z_\sharp(\mathbb{R}^n + i\Gamma_0) \cap B(0, r) \Subset \Omega \Subset Z_\sharp(\mathbb{R}^n + i\widetilde{\Gamma}) \cap B(0, r) .$$

So if we let $\widetilde{\Theta} = Z_\sharp^{-1}(B(0, r)) \subset \Theta$, then

$$W(\Gamma_0, \widetilde{\Theta}) \Subset \Omega \Subset W(\widetilde{\Gamma}_0, \Theta) .$$

$\overline{\partial} f_0$ is a $\overline{\partial}$ closed $(0, 1)$ form on Ω. $\overline{\partial} f_0$ is certainly C^∞ up to the boundary of Ω away from the edge $E(\Theta)$, and we now show $\overline{\partial} f_0$ is also C^∞ up to the edge

$$\overline{\partial} f_0(w) = \sum_{k=1}^{n} \frac{\partial f_0}{\partial \overline{w}_k}(w)\, d\overline{w}_k .$$

So the pullback $(\overline{\partial} f_0)^*$ to $E(\Theta)$ is

$$(\overline{\partial} f_0)^* = \sum_{k=1}^{n} \frac{\partial f_0}{\partial \overline{w}_k}(Z_\sharp(z))\, d\overline{Z}_\sharp^k(z) = \sum_{k=1}^{n} \frac{\partial f_0}{\partial \overline{w}_k}(Z_\sharp(z))(\partial \overline{Z}_\sharp^k(z) + \overline{\partial} \overline{Z}_\sharp^k(z)) .$$

Now $g(z) = f(Z_\sharp(z))$ is holomorphic so

$$0 = \overline{\partial} g = \sum_{k=1}^{n} \frac{\partial f_0}{\partial w_k}(Z_\sharp(z))(\overline{\partial} Z_\sharp^k(z)) + \sum_{k=1}^{n} \frac{\partial f_0}{\partial \overline{w}_k}(Z_\sharp(z))(\overline{\partial} \overline{Z}_\sharp^k(z)) .$$

So we make a substitution into the equation for $(\overline{\partial} f_0)^*$ to obtain

$$(\overline{\partial} f_0)^* = \sum_{k=1}^{n} \frac{\partial f_0}{\partial \overline{w}_k}(Z_\sharp(z))(\partial \overline{Z}_\sharp^k(z)) - \sum_{k=1}^{n} \frac{\partial f_0}{\partial w_k}(Z_\sharp(z))(\overline{\partial} Z_\sharp^k(z)) .$$

We noted before that the coefficients of $\overline{\partial} Z_\sharp^k(z)$ and $\partial \overline{Z}_\sharp^k$ along with all their derivatives vanish to infinite order at $\operatorname{Im} z = 0$ and that the derivatives of f_0 grow at most polynomially at $\operatorname{Im} z = 0$, so we conclude that $(\overline{\partial} f_0)^*$ extends smoothly to the value 0 at $\operatorname{Im} z = 0$. Hence $\overline{\partial} f_0$ is C^∞ in Ω. Therefore, according to Dufresnoy, there exists a function u_0 in Ω such that u_0 is C^∞ up to the boundary of Ω and $\overline{\partial} u_0 = \overline{\partial} f_0$ in Ω. The function $g_0 = f_0 - u_0$ is then a holomorphic function in the smaller wedge $W(\Gamma_0, \widetilde{\Theta})$ and has boundary value on the edge

$$bv\, g_0 = bv\, f_0 + bv\, u_0 \in L^p(E(\widetilde{\Theta})).$$

For each of the other functions $f_1, \ldots, f_\nu$ a similar $\overline{\partial}$ correction can be carried out in appropriate domains. In each case we can write $bv\, f_j = bv\, g_j = bv\, u_j$ on the edge of a smaller wedge $W(\Gamma_j, \widetilde{\Theta})$, where g_j is holomorphic in the wedge having L^p boundary value and $bv\, u_j$ is a C^∞ function on the edge. If we define the C^∞ function u on the edge $E(\widetilde{\Theta})$ by $u = -\sum_{j=0}^{\nu} u_j$ then we have shown there exists a neighborhood $\widetilde{\Theta} \Subset \Theta$ of 0 in $\mathbb{C}^n$ such that every $f \in L^p(E(\Theta))$ can be written on $E(\widetilde{\Theta})$ as

$$f = \sum_{j=0}^{\nu} bv\, g_j + u$$

where for $j = 0, 1, \ldots, \nu$, $bv\, g_j \in L^p(E(\widetilde{\Theta}))$ is the boundary value of a holomorphic function in the wedge $W(\Gamma_j, \widetilde{\Theta})$ and $u \in C^\infty(E(\widetilde{\Theta}))$.

Step 3. Use Theorem 1 to split u. We now use Theorem 1, which states there exists a neighborhood $\Theta' \subset \widetilde{\Theta}$ such that on $E(\Theta')$ we can write $u = \sum_{j=0}^{\nu} bv\, u_j$ where for $j = 0, 1, \ldots, \nu$, $bv\, u_j \in C^\infty(E(\Theta'))$ is the boundary value of a holomorphic function in the wedge $W(\Gamma_j, \Theta')$. Theorem 2 has therefore been proven using this Θ' and taking $f_j = g_j + u_j$ for $j = 0, 1, \ldots, \nu$.

Remarks. To get the L^p decomposition it is not necessary to assume the manifold to be C^∞ but require only smoothness C^k for some k large enough, however, we have not done this. Apply this method to the case $n = 1$, of a curve in $\mathbb{C}$, we have been able to prove the L^p decomposition assuming the curve to have smoothness $C^{1+\alpha}$ for any $\alpha > 1/p$.

3. C^∞ SPLITTING

In this section we introduce the FBI transform as found in [1] and use it to prove Theorem 1. M, $Z = x + i\phi$, and $Z_\sharp$ remain as defined in the previous section.

Definition. Let U be a neighborhood of 0 in $\mathbb{R}^n$ and $u \in C_0^\infty(U)$. Then the *FBI transform* of u is

$$F(u; z, t, \zeta) = \int_{\mathbb{R}^n} e^{i\zeta \cdot [z - Z(y)] - \langle\zeta\rangle [Z(t) - Z(y)]^2} u(y)\, dZ(y)$$

where $t \in \mathbb{R}^n$, $z \in \mathbb{C}^n$, $\zeta \in \mathbb{C}^n$, $|\operatorname{Im} \zeta| < |\operatorname{Re} \zeta|$, $\langle\zeta\rangle = (\zeta_1^2 + \cdots + \zeta_n^2)^{1/2}$, and $[Z(t) - Z(y)]^2 = (Z_1(t) - Z_1(y))^2 + \cdots + (Z_n(t) - Z_n(y))^2$.

Let Γ and C be open cones in $\mathbb{R}^n \backslash \{0\}$ such that $C \Subset \Gamma^0$. Let $d > 0$ and

Θ a neighborhood of 0 in $\mathbb{C}^n$. Then the following restrictions will apply:

(1) $|Z(t)| < d$,
(2) $z \in W(\Gamma, \Theta) = \{Z_\sharp(x+iy)\,;\, x+iy \in (\mathbb{R}^n + i\Gamma) \cap \Theta\}$,
(3) $\zeta = {}^tZ_x(t)^{-1}\xi + 4i\frac{1}{328}|\xi|[z - Z(t)]$ where $\xi \in C$.

Lemma 1 (fast decay of the FBI transform). *If U, d and Θ are small enough and $u \in C_0^\infty(U)$, then for every $m \in \mathbb{N}$ there exists a constant C_m such that $|F(u;\, z, t, \zeta)| < C_m/|\zeta|^m$ where C_m is independent of t and z provided they satisfy the restrictions* (1)–(2) *and ζ satisfies* (3).

Proof. According to [1, pp. 362–365], U, d, and Θ can be made small enough so that $\mathrm{Re}(i\zeta\cdot[z - Z(y)] - \langle\zeta\rangle[Z(t) - Z(y)]^2) < 0$ for all t, z, and ζ that satisfy the restrictions (1)–(3) and $y \in U$. For our purposes we also assume that d and U are small enough such that $|Z(t)| < d$ and $y \in U$ imply $|Z(t) - Z(y)| < 1/(4\sqrt{n})$. There exists vector fields $M_1, \dots, M_n$ such that $M_j(Z_k(y)) = \delta_{jk}$ for all $j, k = 1, \dots, n$. If (a_{jl}) is the inverse matrix of the Jacobian of $Z(y)$ then for $j = 1, \dots, n$, $M_j = \sum_{l=1}^n a_{jl}(\partial/\partial y_l)$. If $u \in C_0^\infty(U)$ and $v \in C^\infty(U)$, then $du = \sum_{j=1}^n M_j(u)\,dZ_j(y)$ and, therefore, we have the following integration by parts formula

$$\int_{\mathbb{R}^n} vM_j(u)\,dZ(y) = -\int_{\mathbb{R}^n} M_j(v)u\,dZ(y) \quad \text{for } j = 1, \dots, n.$$

Fix $\zeta \neq 0$. For some $k \in \{1, \dots, n\}$, $|\zeta_k| \geq |\zeta_j|$ for $j = 1, \dots, n$. Then $|\langle\zeta\rangle| \leq |\zeta| \leq \sqrt{n}|\zeta_k|$. Let

$$S = i\zeta\cdot[z - Z(y)] - \langle\zeta\rangle[Z(t) - Z(y)]^2.$$

Then

$$\begin{aligned} M_k(e^S) &= e^S\{-i\zeta_k + 2\langle\zeta\rangle[Z_k(t) - Z_k(y)]\} \\ &= \zeta_k e^S\left\{-i + \frac{2\langle\zeta\rangle}{\zeta_k}[Z_k(t) - Z_k(y)]\right\} = \zeta_k e^S T, \end{aligned}$$

$$|T| \geq 1 - 2\sqrt{n}|Z_k(t) - Z_k(y)| \geq 1 - 2\sqrt{n}\frac{1}{4\sqrt{n}} \geq \frac{1}{2},$$

which implies, in particular, that in the y variable $1/T \in C^\infty(U)$. Therefore

$$\begin{aligned} F(u;\, z, t, \zeta) &= \int_{\mathbb{R}^n} e^S u(y)\,dZ(y) = \int_{\mathbb{R}^n} (e^S\zeta_k T)\frac{u(y)}{\zeta_k T}\,dZ(y) \\ &= \frac{1}{\zeta_k}\int_{\mathbb{R}^n} M_k(e^S)\frac{u(y)}{T}\,dZ(y) = \frac{-1}{\zeta_k}\int_{\mathbb{R}^n} e^S M_k\left(\frac{u(y)}{T}\right)\,dZ(y), \end{aligned}$$

where we have used the integration by parts formula once. If we use integration by parts m times we obtain

$$F(u;\, z, t, \zeta) = \left(\frac{-1}{\zeta_k}\right)^m \int_{\mathbb{R}^n} e^S M_k\left[\frac{1}{T}M_k\left[\cdots\left[\frac{1}{T}M_k\left(\frac{u(y)}{T}\right)\right]\cdots\right]\right]\,dZ(y),$$

where M_k appears m times in the integral. The term

$$M_k\left[\frac{1}{T}M_k\left[\cdots\left[\frac{1}{T}M_k\left(\frac{u(y)}{T}\right)\right]\cdots\right]\right]$$

can be expanded into a finite sum of terms of the form

$$\text{(const)}\left(\frac{1}{T}\right)^r M_k^s(u)M_k^q(T)$$

where r, s, and q are positive integers. Each of the terms is bounded independent of t and z. Indeed $|(1/T)^r| \le 2^r$ and $|M_k^s(u)| \le \text{(const)} \cdot \sup\{|u^{(\alpha)}(y)|;\, y \in U,\ 0 \le |\alpha| \le s\}$ and $M_k(T) = 2\langle\zeta\rangle/\zeta_k$ so $|M_k(T)| \le 2\sqrt{n}$, and for $q > 1$, $M_k^q(T) = 0$. Therefore

$$M_k\left[\frac{1}{T}M_k\left[\cdots\left[\frac{1}{T}M_k\left(\frac{u(y)}{T}\right)\right]\cdots\right]\right] \le \widetilde{C}_m$$

where $\widetilde{C}_m$ is a constant that depends only on the derivatives of u of order $\le m$. Using the fact that $\operatorname{Re} S < 0$ we conclude that

$$|F(u;\, z,\, t,\, \zeta)| \le \frac{\widetilde{C}_m}{|\zeta_k|^m}\int_U d|Z(y)| \le \frac{(\sqrt{n})^m\widetilde{C}_m}{|\zeta|^m}\int_U d|Z(y)| \le \frac{C_m}{|\zeta|^m}.$$

We now prove Theorem 1.

Theorem 1 (C^∞ splitting). *Let $\Gamma_0, \Gamma_1, \dots, \Gamma_\nu, C_0, C_1, \dots, C_\nu$ be a collection of nonempty open cones such that $C_j \Subset \Gamma_j^0$ for $j = 0, 1, \dots, \nu$, and the C_j cover $\mathbb{R}^n\backslash\{0\}$. Let Θ be an open neighborhood of 0 in C^n. Then there exists an open neighborhood Θ' of 0 such that $\Theta' \subset \Theta$ and every $u \in C^\infty(E(\Theta))$ can be written on $E(\Theta')$ as $u = \sum_{j=0}^{\nu} bvu_j$ where for $j = 0, 1, \dots, \nu$, $bvu_j \in C^\infty(E(\Theta'))$ is the boundary value of a holomorphic function u_j in the wedge $W(\Gamma_j, \Theta')$ and is C^∞ up to the edge $E(\Theta')$.*

Proof of Theorem 1. Let $d > 0$, and $U = \Theta \cap \mathbb{R}^n$. Since this is a local result, we may assume Θ to be as small as we wish. First assume that d, Θ, and, therefore, U are small enough so that Lemma 1 applies. We identify u and its pullback to U (i.e., $u(y) = u(Z(y))$ for $y \in U$). Now cut u off by a function $\chi(y) \in C_0^\infty(U)$ such that $\chi \equiv 1$ in a neighborhood of 0. If we prove the Theorem for χu then it is true for u since $\chi u \equiv u$ in a neighborhood of 0, so without loss of generality, we assume $u \in C_0^\infty(U)$. To use the results in [1] we do not need the C_j to cover $\mathbb{R}^n\backslash\{0\}$, instead we will deal with a collection of open subcones $C_j' \subset C_j$ such that $C_j' \cap C_k' = \varnothing$ if $j \ne k$, $j, k = 0, \dots, \nu$ and $\mathbb{R}^n\backslash(C_0' \cup C_1' \cup \cdots \cup C_\nu')$ has measure zero. We next assume d, Θ, and U are small enough to use Theorem 2.2 in [1] that gives a holomorphic decomposition ($u = \sum u_j$; u_j holomorphic in wedges) of u modulo a function holomorphic in a neighborhood of the origin by setting

$$u_j(z) = \int_{|Z(t)|<d}\int_{\xi\in C_j'} \langle\zeta\rangle^{n/2} F(u;\, z,\, t,\, \zeta)\, d\zeta\, dZ(t)$$

for $j = 0, \dots, \nu$, where $\zeta = {}^tZ_x(t)^{-1}\xi + 4i\frac{1}{328}|\xi|[z - Z(t)]$ for $\xi \in C_j'$, z is restricted to the wedge $W(\Gamma_j, \Theta')$ and Θ' is any open set with compact closure contained in Θ.

Now consider the function $u_0(z)$ that is holomorphic in the wedge $W(\Gamma_0, \Theta')$. We show u_0 is C^∞ up to the edge $E(\Theta')$ by showing that all its derivatives in the z variables ($\overline{z}$ derivatives are 0 since u_0 is holomorphic) are continuous up to the edge. Substituting for $F(u; z, t, \zeta)$ we have

$$u_0(z) = \int_{|Z(t)|<d} \int_{\xi \in C_0'} \langle\zeta\rangle^{n/2} \int_{\mathbb{R}^n} e^{i\zeta \cdot [z-Z(y)] - \langle\zeta\rangle[Z(t)-Z(y)]^2} u(y)\, dZ(y)\, d\zeta\, dZ(t)$$

where

(3) $\zeta = {}^tZ_x(t)^{-1}\xi + B|\xi|[z - Z(t)]$

for $\xi \in C_0'$ and $B = 4i/328$. Assume that d and Θ' are small enough so that if $|Z(t)| < d$, $z \in Z_\sharp(\Theta')$, and ζ is given by (3) then $\frac{1}{2}|\zeta| < |\xi| < 2|\zeta|$ and $\frac{1}{2}|\langle\zeta\rangle| < |\xi| < 2|\langle\zeta\rangle|$. To compute derivatives of $u_0(z)$ we differentiate terms under the integral sign that involve z and ζ. The following formulas are needed:

(F1) $$\frac{\partial}{\partial z_k}\zeta_j = B|\xi|\delta_{jk} \quad \text{where } \delta_{jk} \text{ is the Kronecker delta},$$

(F2) $$\frac{\partial}{\partial z_k}\langle\zeta\rangle = \frac{\partial}{\partial z_k}\sqrt{\zeta_1^2 + \cdots + \zeta_n^2} = \frac{B|\xi|\zeta_k}{\langle\zeta\rangle}.$$

Differentiating u repeatedly leads to expressions of the form

$$\iint P(\zeta)Q(\xi)R(t)S(z) \int e^{i\zeta \cdot [z-Z(y)] - \langle\zeta\rangle[Z(t)-Z(y)]^2} Z(y)^\alpha u(y)$$
$$= \int_{|Z(t)|<d} \int_{\xi \in C_0'} P(\zeta)Q(\xi)R(t)S(z)F(Z^\alpha u; z, t, \zeta)\, d\zeta\, dZ(t)$$

where $R(t)$ is a bounded continuous function on $\{|Z(t)| < d\}$, $S(z)$ is a polynomial, α is a multi-index, and $P(\zeta)$ and $Q(\xi)$ are continuous functions whose derivatives grow at most polynomially in $|\zeta|$.

The polynomial growth of $P(\zeta)Q(\xi)$ is offset by the fast decay of the FBI transform of the function $Z^\alpha u$ as proven in Lemma 1 and, therefore, the integral converges absolutely for all z in the wedge up to the edge and so defines a continuous function up to the edge. This concludes the proof of the C^∞ splitting.

Acknowledgment

I would like to thank Jean-Pierre Rosay for his insights into this problem.

References

1. M. S. Baouendi, C. H. Chang, and F. Treves, *Microlocal hypoanalyticity and extension of CR functions*, J. Differential Geom. **18** (1983), 331–391.
2. A. Dufresnoy, *Sur l'operateur d'et les fonctions differentiables au sens de Whitney*, Ann. Inst. Fourier, Grenoble **29** (1979), 229–238.
3. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, NJ, 1970.

Department of Mathematics, Florida International University, Miami, Florida 33199-0001

E-mail address: calbeckw@serss0.fiu.edu