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# $A_2$-ANNIHILATED ELEMENTS IN $H_*(\Omega\Sigma RP^2)$

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ABSTRACT. Let $X^n$ denote the smash product of $n$ copies of $\mathbb{RP}^2$. We describe a minimal set of generators for $H^*(X^n; \mathbb{Z}_2)$ as a module over the mod 2 Steenrod algebra. The description includes a procedure to obtain all of the generators, a generating function to enumerate them, and a proof of a nice conjecture about how many there are in each dimension.

## 1. INTRODUCTION AND STATEMENT OF THE PRINCIPAL RESULT

Let $P^q = H^*(RP^\infty)^{\otimes q}$ be the mod 2 cohomology of the product of $q$ copies of $RP^\infty$, considered as an $A_2$-module. This topic has been studied much in recent years. Its homological properties have been studied by Lannes and his school, following the work of Carlsson and Miller. It is known to be decomposable, and the indecomposable pieces have been studied by Mitchell and Priddy [6], Harris and Kuhn [5], Campbell and Selick [3], and Harris, Hunter, and Shank [4]. Its relation to $\mathrm{Tor}^{A_2}(Z_2, Z_2)$ has been studied by Singer [8], and its relation to modular representation theory has been studied by Wood [9] and others. To solve the problems raised in [8] and [9], one needs to know a minimal generating set for $P^q$ as an $A_2$-module. A qualitative conjecture as to the structure of this set was given by Peterson in [7], and this was proved by Wood in [10]. However, a more quantitative answer is needed for the applications. By duality it is enough to compute the $A_2$-annihilated elements in $\overline{H}_*(RP^\infty)^{\otimes q}$, or by summing over $q$ it is enough to find $\mathrm{ann}_{A_2} H_*(\Omega\Sigma RP^\infty) \cong \mathrm{ann}_{A_2}(\bigoplus_q \overline{H}_*(RP^\infty)^{\otimes q})$. This seems to be a difficult problem.

The problem of finding $\mathrm{ann}_{A_2} H_*(\Omega\Sigma RP^2)$ is clearly a related problem to $\mathrm{ann}_{A_2} H_*(\Omega\Sigma RP^\infty)$. This problem turns out to have a concisely expressable answer, whose exposition is the purpose of this paper. The complexity and the combinatorics involved in the solution at least suggest what the solution for $\Omega\Sigma RP^\infty$ might look like. The indecomposables we find for $H^*(RP^2)^{\otimes q}$ must remain indecomposable in $H^*(RP^\infty)^{\otimes q}$, so our answer gives a crude lower bound on the number of generators in each dimension. As a final motivation, the problem of determining the $A_2$-module generators for $H^*(RP^2)^{\otimes q}$ is a reasonably natural question in its own right, and in the final section we note

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that it is equivalent to the corresponding question for certain other spaces as well.

We proceed to give several paragraphs of background and easy definitions; then we state our main result as Theorem 1.

Throughout the calculation, $F$ denotes the field of integers modulo two, and $\langle a, b\rangle$ denotes the free monoid on the two symbols $a$ and $b$. The free $F$-algebra on $a$ and $b$ has $\langle a, b\rangle$ as an $F$-basis and is denoted $F\langle a, b\rangle$ or $T\langle aF \oplus bF\rangle$, the latter notation referring to its interpretation as the tensor algebra on a two-dimensional vector space. We identify $H_*(\Omega\Sigma RP^2)$ with $F\langle a, b\rangle$ as algebras by letting $a$ and $b$ be the images of the generators of $H_1(RP^2)$ and of $H_2(RP^2)$, respectively, under the identity adjoint $RP^2 \to \Omega\Sigma RP^2$. Thus $Sq_*^1(b) = a$, $Sq_*^k(b) = 0$ for $k > 1$, and $Sq_*^k(a) = 0$ for $k > 0$.

The monoid $\langle a, b\rangle$ is viewed as a submonoid of $F\langle a, b\rangle$ and its elements are called *monomials*. Any right factor of a monomial $x$ is called a *tail* of $x$. We impose the lexicographic ordering on $\langle a, b\rangle$, and we call one monomial *lower* than another if it would appear earlier in the dictionary. The *high term* of a nonzero sum of distinct monomials $w \in F\langle a, b\rangle$ is the lexicographically highest contributor to the sum and is denoted $w'$. Notice that $(uv)' = u'v'$. The *length* of a monomial is the total number of symbols (the number of $a$'s plus the number of $b$'s) while its *excess*, denoted $e(\ )$, is the number of $b$'s minus the number of $a$'s. An element of $F\langle a, b\rangle$ is called *bihomogeneous* if it is a sum of monomials sharing a uniform length and a uniform excess.

For simplicity we denote the vector space span of all bihomogeneous $Sq_*$-invariant elements of $F\langle a, b\rangle$ by $A$. We allow only bihomogeneous elements since these are the elements that correspond to a fixed topological dimension and to a fixed number $q$ of factors in the product $(RP^2)^{\wedge q}$. Notice that $A$ is a subalgebra of $F\langle a, b\rangle$; by [1] it is necessarily a tensor algebra.

An essential ingredient in our analysis of $A$ will be the homomorphism $\theta\colon F\langle a, b\rangle \to F\langle a, b\rangle$ given by $\theta(w) = bwa + awb$, which increases length by two but does not affect the excess. Define $S$ to be the least subalgebra of $F\langle a, b\rangle$ that contains 1 and is closed under $\theta$. Let $\theta'\colon \langle a, b\rangle \to \langle a, b\rangle$ be given by $\theta'(x) = bxa$, and let $S'$ be the least submonoid of $\langle a, b\rangle$ containing 1 and closed under $\theta'$. We shall see shortly that $S'$ consists of the high terms of the nonzero elements of $S$.

Our principal result is

**Theorem 1.** *$A$ is the tensor algebra on $\theta(S) \oplus aF$; in particular, it is generated by $S \cup \{a\}$.*

In §2, we give the proof of this result; in §3 we count the size of $A$; and in §4 we give a basis for $S$ and describe $\operatorname{ann}_{A_2} H_*(\Omega\Sigma RP^3)$.

## 2. Proof of Theorem 1

After ten fairly easy lemmas of a combinatorial nature, the proof of Theorem 1 will be nearly a triviality.

**Lemma 2.1.** *A monomial $x$ belongs to $S'$ if and only if $e(x) = 0$ and every tail of $x$ has nonpositive excess.*

*Proof.* Let $S_0'$ denote the set of monomials $x$ having the latter property. Clearly $S' \subset S_0'$, since $S_0'$ is a submonoid of $\langle a, b\rangle$ closed under $\theta'$. Let

$x \in S_0'$ and assume inductively that words in $S_0'$ of length shorter than $x$ are known to belong to $S'$. There is a unique maximal factorization of $x$ as $x = x_1 \cdots x_m$ in $\langle a, b\rangle$ where each $x_j$ has zero excess ("maximal" means that the number $m$ of factors is maximal). Each $x_j$ clearly belongs to $S_0'$. If $m > 1$ we are done because the inductive hypothesis gives that $x_j \in S'$. If $m = 1$, then every proper tail of $x$ has negative excess, and it follows that $x = bya$ with $y \in S_0'$. So $y \in S'$ and then $x \in S'$, as desired.

**Lemma 2.2.** *$S'$ equals the free monoid on $\theta'(S')$.*

*Proof.* We must show that each $x \in S' - \{1\}$ has a unique factorization as $\theta'(x_1)\theta'(x_2)\cdots\theta'(x_m)$ with $x_j \in S'$. The maximal factorization described above provides this.

**Lemma 2.3.** *There exists a multiplicative injection $f\colon S' \to S$ such that $f(S')$ is an $F$-basis for $S$ and $(f(x))' = x$ for any $x \in S'$.*

*Proof.* Put $f(1) = 1$. Assuming $f$ has been defined and has the stated properties on words shorter than $x \in S'$, factor $x$ uniquely as $x = x_1 \cdots x_m$ where each $x_j \in \mathrm{Im}(\theta')$. If $m > 1$, put $f(x) = f(x_1)\cdots f(x_m)$. If $m = 1$, write $x = \theta'(y)$ and put $f(x) = \theta(f(y))$. The reader can now check that $(f(x))' = x$. Since $f(S')$ is closed under $\theta$, we have $S = f(S')F$, whence $f(S')$ is an $F$-basis.

**Lemma 2.4.** *$S$ coincides with the tensor algebra on $\theta(S)$.*

*Proof.* Immediate from Lemmas 2.2 and 2.3.

Let $S_+$ denote the subalgebra of $F\langle a, b\rangle$ generated by $S$ and $\{b\}$ and let $S_-$ denote the subalgebra generated by $S$ and $\{a\}$. Let $S_+'$ (resp. $S_-'$) be the submonoid of $\langle a, b\rangle$ generated by $S'$ and $\{b\}$ (resp. $S'$ and $\{a\}$). Later we will show that $A = S_-$.

**Lemma 2.5.** *The following four sets coincide:* (i) $S_+'$; (ii) *the free monoid on $\theta'(S') \cup \{b\}$*; (iii) *the set of high terms of nonzero elements of $S_+$*; (iv) *the set $\{x \in \langle a, b\rangle | e(x) \geq 0$ and every tail of $x$ has excess $\leq e(x)\}$. Furthermore, $f$ extends to $f_+\colon S_+' \to S_+$ such that $f(S_+')$ is an $F$-basis and $(f_+(x))' = x$ and $S_+ = T(\theta(S) \oplus bF)$.*

*Proof.* This is like Lemmas 2.1–2.4. The unique maximal factorization of an $x \in S_+'$ is obtained by breaking $x$ at every point that yields a tail $z$ whose excess is greater than or equal to that of all the tails of $z$.

**Lemma 2.6.** *The following four sets coincide:* (i) $S_-'$; (ii) *the free monoid on $\theta'(S') \cup \{a\}$*; (iii) *the set of high terms of nonzero elements of $S_-$*; (iv) *the set $\{x \in \langle a, b\rangle | x$ and every tail of $x$ has nonpositive excess$\}$. Furthermore, $f$ extends to $f_-\colon S_-' \to S_-$ such that $f(S_-')$ is an $F$-basis and $(f_-(x))' = x$ and $S_- = T(\theta(S) \oplus aF)$.*

*Proof.* This is dual to that of Lemma 2.5.

**Lemma 2.7.** *$Sq_*^k(w) = 0$ if $k > 0$ and $w \in S_-$, i.e., $S_- \subset A$.*

*Proof.* $A$ is easily seen to be a subalgebra of $F\langle a, b\rangle$ that is closed under $\theta$, so $S \subset A$. Since $a \in A$, $S_- \subset A$.

**Lemma 2.8.** *Let $S_k$ denote the subset of $S_+$ (resp. $S_-$) consisting of elements of excess $k$ if $k \geq 0$ ( resp. $k \leq 0$). For $k \geq 0$, $Sq_*^k\colon S_k \to S_{-k}$ is bijective* *and* $Sq_*^{>k}(S_k) = 0$. *In particular, if* $w \neq 0$ *and* $w$ *is bihomogeneous of excess* $k$, *then* $Sq_*^k(w) \neq 0$ *and* $Sq_*^{>k}(w) = 0$.

*Proof*. A basis element of $S_k$ may be written $w = w_0 b w_1 b \cdots b w_k$ where $w_j \in f(S')$. Then $Sq_*^{>k}(w) = 0$ while $Sq_*^k(w) = w_0 a w_1 a \cdots a w_k$, which is the corresponding basis element of $S_{-k}$.

**Lemma 2.9.** *Let* $w \in F\langle a, b\rangle$ *be a nonzero bihomogeneous element and suppose* $k = e(w) \geq 0$. *Then there is some* $j \geq k$ *for which* $S^j_{q_*}(w) \neq 0$. *If, in addition,* $Sq_*^{>k}(w) = 0$, *then* $w \in S_+$.

*Proof*. We may assume inductively that the lemma holds for elements shorter than $w$, as well as for elements of the same length as $w$ whose high terms are exeeded by $w'$. Consider two cases.

If $w'$ has any tail $y_0$ whose excess is greater than $k$, write $w' = x_0 y_0$ and $w = x_0 y + v$ where $y' = y_0$ and either $v = 0$ or $v'$ is lower than $x_0$. The inductive hypothesis says that $Sq_*^j(y) \neq 0$ for a suitable $j \geq e(y) \geq k+1$. Then

$$Sq_*^j(w) = x_0 Sq_*^j(y) + (\text{terms lower than } x_0) \neq 0.$$

Clearly, "$Sq_*^{>k}(w) = 0$" is impossible in this case.

If instead every tail of $w'$ has excess $\leq k$, then $w' \in S'_+$ by Lemma 2.5. Let $u = w - f_+(w')$. If $u = 0$, then $w = f_+(w') \in S_+$ and we are done, so suppose $u \neq 0$ and notice that $u' < w'$. By our inductive hypothesis, $Sq_*^j(u) \neq 0$ for some $j \geq k$. If $j$ can be chosen to be larger than $k$, then $Sq_*^j(w) \neq 0$ because (by Lemma 2.8) $Sq_*^{>k}(f_+(w')) = 0$. If instead $j = k$ with $Sq_*^{>k}(u) = 0$, then $u \in S_+$ whence $w \in S_+$ and $Sq_*^k(w) \neq 0$. Conversely, if $Sq_*^{>k}(w) = 0$, then $Sq_*^{>k}(u) = 0$, implying $u \in S_+$ and then $w \in S_+$.

**Lemma 2.10.** *Let* $x \in A$ *be bihomogeneous and nonzero. Then* $x' \in S'_-$.

*Proof*. Using Lemma 2.6 it suffices to check that every tail of $x'$ has nonpositive excess. Suppose not, and write $x' = y_0 z_0$ with $e(z_0) > 0$. Next, write $x = y_0 z + v$ where $z' = z_0$ and either $v = 0$ or $v'$ is lower than $y_0$. By the previous lemma, $Sq_*^j(z) \neq 0$ for some $j > 0$. Then

$$Sq_*^j(x) = y_0 Sq_*^j(z) + (\text{terms lower than } y_0) \neq 0,$$

contradicting the hypothesis $x \in A$.

*Proof of Theorem* 1. By Lemmas 2.6 and 2.7, we need only show that $A \subset S_-$. Let $x \in A$ be bihomogeneous nonzero, and assume that elements of $A$ with high term exceeded by $x'$ have already been shown to belong to $S_-$. Since $x' \in S'_-$, we may put $u = x - f_-(x')$. If $u = 0$, we are done. If $u \neq 0$, then $u'$ is lower than $x'$; but $u \in A$, so $u \in S_-$. Then $x \in S_-$.

## 3. Counting $A$

In this section we give an explicit formula for the size of a basis for $A$. The following theorem was conjectured by W. S. Wilson and it gave us a good start on our results.

**Theorem 3.1.** *Suppose* $q \geq 2m \geq 0$. *The number of linearly independent bihomogeneous elements in* $A$ *of length* $q$ *and excess* $2m-q$ *is* $\binom{q}{m} - \binom{q}{m-1}$. *This is* *also the number of minimal generators for* $H^*((RP^2)^{\wedge q})$ *as an* $A_2$*-module that occur in topological dimension* $q+m$. *The total number of minimal generators for* $H^*((RP^2)^{\wedge q})$ *is* $\binom{q}{[q/2]}$.

In order to prove Theorem 3.1, we employ generating functions. Since these generating functions will involve the Catalan sequence $\{C_m\}$, we provide a quick review of some of its properties. The $m$th Catalan number is defined by $C_m = \binom{2m}{m}/(m+1)$.

**Lemma 3.2.** *Each of the following properties is satisfied by and uniquely determines the Catalan sequence*:

(a) $C_k = \binom{2k}{k} - \binom{2k}{k-1}$, $k \geq 0$;
(b) $C_p = \binom{2p-1}{p-1} - \binom{2p-1}{p-2}$ *for* $p \geq 1$, *and* $C_0 = 1$;
(c) $C_p = \sum_{n=1}^{p} C_{n-1}C_{p-n}$ *if* $p > 0$, *and* $C_0 = 1$;
(d) *If* $C(x) = \sum_{n=0}^{\infty} C_n x^n$ *denotes the generating function, then* $xC(x)^2 + 1 = C(x)$.

**Lemma 3.3.** *The number of basis elements of* $S$ *of length* $2n$ *is* $C_n$.

*Proof.* If $s_n$ denotes this number, then the generating function of $\{s_n\}$ is $S(x) = \sum_{n=0}^{\infty} s_n x^{2n}$. Since $\theta$ raises length by two, the corresponding generating function for $\theta(S)$ is $x^2S(x)$. By Lemma 2.4 we have $(1 - x^2S(x))^{-1} = S(x)$, which gives $x^2S(x)^2 + 1 = S(x)$. By Lemma 3.2(d), we must have $S(x) = C(x^2)$, whence $s_n = C_n$.

We proceed next with two combinatorial identities.

**Lemma 3.4.** *If* $r \geq 2s$,

$$\sum_{n=1}^{[r/2]} C_{n-1}\left(\binom{r-2n}{s-n} - \binom{r-2n}{s-n-1}\right) = \binom{r-1}{s-1} - \binom{r-1}{s-2}.$$

*Proof.* Denote the left- and right-hand sides by $D(r, s)$ and $E(r, s)$, respectively. Notice that Lemma 3.2(b) says $C_p = E(2p, p)$. An easy consequence of 3.2(a), (c) (put $k = p - n$) is

$$D(2p, p) = E(2p, p) \quad \text{for } p > 0. \tag{1}$$

To prove the lemma, first notice that

$$\begin{aligned} E(r, s) &= E(r-1, s) + E(r-1, s-1) \quad \text{if } s > 0, \\ D(r, s) &= D(r-1, s) + D(r-1, s-1) \quad \text{if } r > 2s. \end{aligned} \tag{2}$$

Let $P(s)$ be the proposition "$D(r, s) = E(r, s)$ for all $r \geq 2s$".

Now $P(0)$ is true, because both quantities are zero. Let $s > 0$ and assume $P(s-1)$; then the equation $D(r, s) = E(r, s)$ follows by induction on $r$, using (1) as the intial step and (2) as the inductive step, so $P(s)$ is true. Thus $P(s)$ is true for all $s \geq 0$.

**Lemma 3.5.** *The following is an identity in power series in two variables*:

$$\left(\sum_{q=0}^{\infty}\sum_{m=0}^{[q/2]}\left(\binom{q}{m} - \binom{q}{m-1}\right)x^q y^{q-2m}\right)\left(1 - xy - \sum_{n=1}^{\infty} C_{n-1}x^{2n}\right) = 1.$$

*Proof.* When any nonzero term in the expansion of the left-hand side is written as a coefficient times $x^r y^{r-2s}$, notice that $r \geq 2s$. This coefficient is

$$\begin{aligned}\left(\binom{r}{s} - \binom{r}{s-1}\right) - \left(\binom{r-1}{s} - \binom{r-1}{s-1}\right)\\ - \sum_{n=1}^{\infty} C_{n-1}\left(\binom{r-2n}{s-n} - \binom{r-2n}{s-n-1}\right).\end{aligned} \tag{3}$$

The first two terms of (3) combine to $\binom{r-1}{s-1} - \binom{r-1}{s-2}$ whenever $r > 0$. Since we may assume that $r \geq 2s$, Lemma 3.4 shows that (3) vanishes except when $r = s = 0$, in which case it is $\binom{0}{0} = 1$.

*Proof of Theorem* 3.1. Let $\lambda_{qk}$ be the dimension of the vector space spanned by bihomogeneous elements of $A$ having length $q$ and excess $-k$. Denote its two-variable generating function by

$$L(x, y) = \sum \lambda_{qk} x^q y^k.$$

By Theorem 1, this generating function coincides with that of the tensor algebra on $aF \oplus \theta(S)$, a vector space whose basis generating function is

$$xy + x^2 S(x) = xy + \sum_{n \geq 1} C_{n-1} x^{2n}.$$

Thus

$$L(x, y)^{-1} = 1 - xy - \sum_{n=1}^{\infty} C_{n-1} x^{2n}.$$

By Lemma 3.5 we have

$$L(x, y) = \sum_{q=0}^{\infty} \sum_{m=0}^{[q/2]} \left(\binom{q}{m} - \binom{q}{m-1}\right) x^q y^{q-2m}.$$

Thus $\lambda_{qk} =$ either $\binom{q}{m} - \binom{q}{m-1}$ if $-k = 2m - q$, or 0 if $k < 0$ or $k \not\equiv q \pmod 2$. Invariants of length $q$ correspond to $A_2$-module generators for $H^*((RP^2)^{\wedge q})$, and those of excess $2m - q$ involve precisely $m$ $b$'s and $q - m$ $a$'s and, therefore, occur in topological dimension $q + m$.

## 4. Some corollaries

Recalling that the algebra $A$ is free on the set $\theta(f(S')) \cup \{a\}$, let us write down explicitly how $\theta(f(S'))$ is constructed. Let $w = \theta(1) = ab + ba$, and let $V_1 = \{w\}$. Build the free monoid $\langle V_1 \rangle$ on $V_1$, which is the set $\{1, w, w^2, w^3, \ldots\}$, and apply $\theta$ to each element of it. We obtain a set

$$V_2 = \{w, bwa + awb, bw^2 a + aw^2 b, \ldots\}$$

that contains $V_1$. Having constructed $V_1 \subset V_2 \subset \cdots \subset V_t$, let $\langle V_t \rangle$ be the free monoid on $V_t$ and let $V_{t+1} = \theta(\langle V_t \rangle)$. The following is an easy corollary of the construction.

**Corollary 4.1.** $\theta(f(S')) = \bigcup_{t>0} V_t$.

Let $\mathcal{S}_t$ denote the symmetric group on $t$ letters. It acts on the vector space of bihomogeneous elements of length $t$ and excess $k$ in $F\langle a, b\rangle$. For $x \in F\langle a, b\rangle$ of length $t$ and for any $\sigma \in \mathcal{S}_t$, notice that $Sq_*^k(x) = 0$ if and only if $Sq_*^k(\sigma(x)) = 0$. The permutation that cycles the first $t+1$ symbols will convert $xw$ to $\theta(x)$ when $x$ has length $t$. Using these facts we obtain the following description of $A$.

**Corollary 4.2.** *Let $x \in F\langle a, b\rangle$ be bihomogeneous of excess $-k$ $(k \geq 0)$ and length $2m+k$. Then $x \in A$ if and only if there exists $\sigma \in \mathcal{S}_{2m+k}$ such that $x = \sigma(w^m a^k)$.*

The proof of Theorem 1 used only the facts that our tensor algebra had two generators connected by a $Sq_*^1$ and that $Sq_*$ satisfies the Cartan formula. The proof will also work for other tensor algebras having two generators connected by a $Sq_*^k$.

**Corollary 4.3.** *The proof of Theorem 1 works equally well to compute $\operatorname{ann}_{A_2} H_*(\Omega\Sigma X)$, where $X = CP^2$, $HP^2$, or the Cayley projective plane.*

Finally, we note that $H_*(\Omega\Sigma RP^3)$ can be handled without incurring additional difficulties. Letting $c$ denote the nonzero element in the image of $H_3(RP^3)$ in $H_3(\Omega\Sigma RP^3)$, we have $H_*(\Omega\Sigma RP^3) = F\langle a, b, c\rangle$. Because $Sq_*^{>0}(c) = 0$, $c$ can be "inserted into" or "deleted from" any element of $F\langle a, b, c\rangle$ without altering whether or not that element lies in $\operatorname{Ker}(Sq_*^k)$. To make this precise, we state

**Lemma 4.4.** *Let $x \in F\langle a, b\rangle$ be bihomogeneous of length $t$. For any $m$ and any $\sigma \in \mathcal{S}_{t+m}$, $\sigma(xc^m) \in \operatorname{ann}_{A_2} H_*(\Omega\Sigma RP^3)$ if and only if $x \in A$.*

Define $\theta\colon F\langle a, b, c\rangle \to F\langle a, b, c\rangle$ by $\theta(x) = bxa + axb$.

**Corollary 4.5.** *Let $V$ denote the least subalgebra of $F\langle a, b, c\rangle$ that contains $c$ and is closed under $\theta$. Then $\operatorname{ann}_{A_2} H_*(\Omega\Sigma RP^3)$ is the tensor algebra on the vector space $\theta(V) \oplus aF \oplus cF$.*

A basis for $\theta(V)$ can be listed as in Corollary 4.1, by starting with the set $\{1, c\}$ and iterating the process of applying $\theta(\langle\ \rangle)$. We leave the details to the reader.

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