PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 117, Number 1, January 1993

DISTINCT 3-MANIFOLDS WITH ALL $\mathrm{SU}(2)_q$ INVARIANTS THE SAME

W. B. R. LICKORISH

(Communicated by James E. West)

ABSTRACT. Witten's $\mathrm{SU}(2)_q$ invariants do not classify 3-manifolds.

The aim of this note is to prove, in two ways, the following result.

Theorem. *There exist pairs* $\{M_1, M_2\}$ *of closed oriented* 3*-manifolds such that* M_1 *and* M_2 *have the same (Witten)* $\mathrm{SU}(2)_q$ *invariants for all roots or unity (for all levels), but such that there is no orientation preserving homeomorphism between them.*

Two methods of producing such pairs will be described; they are motivated by two ways of obtaining links with the same Jones polynomial. The first method consists essentially of gluing together the complements of two nonreversible knots in two different ways. In the second, the complement of one knot is glued to the complements of two other knots that are related by mutation. If the Witten invariants corresponding to Lie groups other than $\mathrm{SU}(2)$ can be described in a similar way to that explained below for $\mathrm{SU}(2)$, if they can be interpreted, via parallels of link diagrams, in terms of the various 2-variable generalisations of the Jones polynomial, then one might expect that at least the first method would provide pairs of 3-manifolds with all Witten invariants the same. Using the same underlying Jones polynomial phenomena, a different collection of pairs of 3-manifolds not distinguished by the $\mathrm{SU}(2)_q$ invariants has been obtained independently and simultaneously by J. Kania-Bartoszynska.

Throughout, manifolds, submanifolds, and homeomorphisms are to be taken as smooth or piecewise linear.

The $\mathrm{SU}(2)_q$ invariants of Witten are described in [14, 15] using the following terminology. Denote by $\mathfrak{A}$ the appropriate linear skein [11, 12] of the standard annulus $\{z \in \mathbb{C}: 1 \leq |z| \leq 2\}$. Thus $\mathfrak{A}$ is the quotient of the $\mathbb{Z}[A, A^{-1}]$-module freely generated by ambient isotopy classes of all link diagrams in the annulus, quotiented by the relations

(i) $X \cup$ (a closed nul-homotopic component with no crossing) $= (-A^{-2} - A^2)X$,

(ii) $\times = A\,)(\, + A^{-1} \asymp$.

Received by the editors May 28, 1991.
1991 *Mathematics Subject Classification.* Primary 57M25; Secondary 57N10, 81R99.

Here X is any link diagram in the annulus and (ii) refers to any three such diagrams identical except where shown. It follows that diagrams regularly isotopic in the annulus represent the same element of $\mathfrak{A}$. If α^i is the element represented by i parallel simple closed curves all encircling the annulus, then $\{\alpha^0, \alpha^1, \alpha^2, \dots\}$ is a base for $\mathfrak{A}$ (α^0 being represented by the empty diagram). Suppose that D is a *planar* diagram of an n-component link. Neighbourhoods of these components may be taken to be n annuli immersed in the plane with over- and under-crossing information preserved from the crossings of D. Consider the operation of taking n link diagrams in n standard annuli, inserting them in the immersed annuli, obeying the over and under crossing instructions in the obvious way, and then evaluating the Kauffman bracket. This operation induces a well defined n-multilinear map

$$\Phi_D\colon \mathfrak{A} \times \mathfrak{A} \times \cdots \times \mathfrak{A} \to \mathbb{Z}[A, A^{-1}].$$

Note that if Δ is an n-component diagram in the annulus similar insertion induces a multilinear map

$$\Psi_\Delta\colon \mathfrak{A} \times \mathfrak{A} \times \cdots \times \mathfrak{A} \to \mathfrak{A}.$$

The operation, of reflecting the annulus $(z \to \overline{z})$ together with changing every over-crossing to an under-crossing, gives a permutation ρ on the diagrams in the annulus that clearly passes to the quotient to induce a linear isomorphism ρ^* of $\mathfrak{A}$. However, as $\rho^*(\alpha^i) = \alpha^i$, ρ^* is the identity map. Of course, if the diagrams are regarded as actual links in a solid torus that is a slight thickening of the annulus, then ρ corresponds to the effect of a π-rotation of the solid torus about an axis meeting the solid torus in two arcs. If Δ is an n-component diagram in the annulus then

$$\Psi_{\rho\Delta} = \rho^*\Psi_\Delta = \Psi_\Delta.$$

Now, suppose the oriented 3-manifold M is obtained by surgery on the n-component framed link L (in S^3) that is represented by a planar diagram D in which the writhe of each component is the framing of the corresonding component of L. Let A be a primitive $4r$th root of unity so that elements of $\mathbb{Z}[A, A^{-1}]$ become evaluated as complex numbers. Corresponding to A there is a (very special) element $\mathbf{a} \in \mathfrak{A}$ so that the complex number

$$(\Phi_{U(-1)}(\mathbf{a}))^{(\sigma+\nu-n)/2}\Phi_D(\mathbf{a}, \mathbf{a}, \dots, \mathbf{a})$$

is the A-$\mathrm{SU}(2)_q$ invariant of M. Here σ and ν are the signature and nullity of the linking matrix of L (with the framings on the diagonal) and $U(-1)$ is the diagram of the unknot with one negative crossing. Details are to be found in [15] where it is shown that this is essentially the same invariant as that of Reshetikhin and Turaev [17, 10]. The algebra of [14, 15] was extended in [1] to allow A to be a $2r$th root of unity; the invariant has exactly the same form but a different $\mathbf{a}$ is used.

Let S^1 be the unit complex numbers and let (-1) be the automorphism of the torus $S^1 \times S^1$ defined by $(-1)(z_1, z_2) = (\overline{z}_1, \overline{z}_2)$. This represents the nontrivial central element of the mapping-class group of $S^1 \times S^1$. In what follows φ will be a homeomorphism from one torus to another; $-\varphi$ will denote the composition $(-1) \circ \varphi = \varphi \circ (-1)$.

Lemma 1. *Suppose X_1 and X_2 are compact connected oriented 3-manifolds each having boundary a torus. Let $\phi\colon \partial X_1 \to \partial X_2$ be an orientation reversing homeomorphism. Then for all primitive $2r$th roots of unity A, $X_1 \cup_\phi X_2$ and $X_1 \cup_{-\phi} X_2$ have the same $\mathrm{SU}(2)_q$ invariants.*

Proof. The manifold $X_1 \cup_\psi (S^1 \times D^2)$, for any homeomorphism $\psi\colon \partial X_1 \to \partial(S^1 \times D^2)$, is obtained by surgery on a framed link in S^3. The solid torus in S^3 corresponding to $S^1 \times D^2$ may be taken to be disjoint from that link and, by further surgery using the technique described in [6], it can be taken to be unknotted. Thus X_1 is obtained by surgery on a solid torus, and, after yet more surgery, ∂X_1 can be identified in any prescribed way with the boundary of that solid torus. Combined with a similar consideration for X_2, this shows that $X_1 \cup_\phi X_2$ can be obtained from S^3 by surgery on a framed link L with the following properties. L is the disjoint union of sublinks L_1 and L_2 contained in the interior of solid tori T_1 and T_2, respectively, where T_1 and T_2 form a Heegaard splitting of S^3 (and are hence unknotted solid tori, linked as a standard Hopf link). Further, each X_i is obtained by surgery of T_i on the framed link L_i. Let H be the standard diagram, of the Hopf link of two components, that has two crossings. Then there are diagrams Δ_1 and Δ_2 in the annulus so that if Δ_1 is inserted around one component of H and Δ_2 around the other then the result is a diagram D for the framed link L (each inserted Δ_i corresponding to L_i). Insertion of Δ_1 and $\rho\Delta_2$ gives a diagram D' of a framed link L', surgery on L' giving $X_1 \cup_{-\phi} X_2$. Then

$$\begin{aligned}\Phi_D(\mathbf{a}, \mathbf{a}, \dots, \mathbf{a}) &= \Phi_H(\Psi_{\Delta_1}(\mathbf{a}, \mathbf{a}, \dots, \mathbf{a}), \Psi_{\Delta_2}(\mathbf{a}, \mathbf{a}, \dots, \mathbf{a}))\\ &= \Phi_H(\Psi_{\Delta_1}(\mathbf{a}, \mathbf{a}, \dots, \mathbf{a}), \Psi_{\rho\Delta_2}(\mathbf{a}, \mathbf{a}, \dots, \mathbf{a}))\\ &= \Phi_{D'}(\mathbf{a}, \mathbf{a}, \dots, \mathbf{a}).\end{aligned}$$

If now L is oriented, it has a linking matrix. If an orientation of L' is chosen to equal that of L restricted to L_1 and to be the reverse of that of L on (its copy of) L_2, then L and L' have equal linking matrices. Thus the manifolds $X_1 \cup_\phi X_2$ and $X_1 \cup_{-\phi} X_2$ have the same A-$\mathrm{SU}(2)_q$ invariants.

The preceding lemma produces pairs of manifolds with the same invariants; it is now necessary to show that sometimes these manifolds are distinct. Doubtless in particular cases that could be proved by analysing the fundamental group, but use of some standard results about 3-manifolds leads to classes of such distinct pairs. Recall that a 3-manifold M is *irreducible* if every 2-sphere embedded in M bounds a 3-ball contained in M and that M is said to be *atoroidal* if every incompressible torus in M is parallel to a component of ∂M (if M is orientable 'incompressible' means that the inclusion map of the torus induces an injection on fundamental groups). The following lemma gives sufficient conditions under which $X_1 \cup_\phi X_2$ and $X_1 \cup_{-\phi} X_2$ are distinct.

Lemma 2. *Suppose that 3-manifolds X_1 and X_2 are compact, connected, oriented, and irreducible, and each has an incompressible torus as its boundary. Suppose, in addition, that one of X_1 and X_2 fails to be Seifert fibred, that both are atoroidal, and that for $i = 1, 2$ if $h\colon X_i \to X_i$ is any orientation preserving homeomorphism then $h|\partial X_i\colon \partial X_i \to \partial X_i$ is isotopic to the identity. Let $\phi\colon \partial X_1 \to \partial X_2$ be an orientation reversing homeomorphism. If* (i) *X_1 and X_2*

are not homeomorphic, or (ii) $X_1 = X_2$ *but* ϕ^2 *is not isotopic to* (-1), *then* $X_1 \cup_\phi X_2$ *and* $X_1 \cup_{-\phi} X_2$ *are not homeomorphic as oriented* 3*-manifolds.*

Proof. Let M and M^- be $X_1 \cup_\phi X_2$ and $X_1 \cup_{-\phi} X_2$, respectively. M and M^- contain incompressible tori T and T^-, namely, the copies of ∂X_1. Neither M nor M^- is Seifert fibred; otherwise the incompressible torus could be taken to be a union of fibres and the fibration would then restrict to a fibration on both X_1 and X_2. Because X_1 and X_2 are atoroidal, the Characteristic Variety Theorem for irreducible 3-manifolds [8, 9] asserts that the incompressible tori T and T^- are unique up to ambient isotopy (only an easy part of that theorem is needed). Thus, if $h: M \to M^-$ is an orientation preserving homeomorphism it may be assumed, after isotopy, that $h(T) = T^-$. In case (i) h must map the 'first half' of M to the 'first half' of M^-. Thus h decomposes as $h_1: X_1 \to X_1$ and $h_2: X_2 \to X_2$ with the compatibility condition that for all x in ∂X_1, $-\phi h_1(x) = h_2\phi(x)$. But $h_1|\partial X_1$ and $h_2|\partial X_2$ are isotopic to the identity, so this implies the false assertion that (-1) is isotopic to the identity map of the torus. In case (ii) there is the additional possibility that perhaps $h_1(X_1) = X_2$ and vice versa. Then $-\phi h_2 \phi(x) = h_1(x)$ for all x in ∂X_1. If then X_1 and X_2 are regarded as equal, $h_1|\partial X_1$ and $h_2|\partial X_2$ are again isotopic to the identity and hence $-\phi^2$ is also isotopic to the identity.

Reassurance is now needed that it is easy to find manifolds satisfying all the conditions of the last lemma. Suppose k is a knot (just a simple closed curve) in S^3. The *exterior* $X(k)$ of k is the complement of the interior of a regular neighbourhood $N(k)$ of k. Then it follows from the Loop and Sphere Theorems (see [7], for example) that the compact connected oriented 3-manifold $X(k)$ is irreducible and, if k is nontrivial then $\partial X(k)$ is incompressible. The knot k is *reversible* in the sense of Conway [4] (or invertible in the sense of Trotter [18]) if there exists an orientation preserving homeomorphism $F: S^3 \to S^3$ such that $F(k) = k$ but F reverses the orientation of k. The existence of nonreversible knots was established by Trotter in [18] where he showed that the pretzel knot (see Figure 1) $K(p, q, r)$ is not reversible provided $|p|$, $|q|$, and $|r|$ are distinct odd integers greater than 1. His technique used a representation of $\pi_1(X(k))$ onto a triangle group of planar hyperbolic isometries. Other nonreversible Montesinos knots are described in [3]. The first nonreversible knot in the tables is 8_{17} (it is not a pretzel knot, see [2]).

A pretzel knot $k = K(p, q, r)$ with $|p|$, $|q|$, and $|r|$ odd integers greater than 1, is *simple* in the Schubert sense; that is equivalent to saying that $X(k)$ is atoroidal. This follows from a consideration of the fact that the double cover of S^3 branched over k is an atoroidal Seifert fibre space.

$K(-3, 5, 7)$

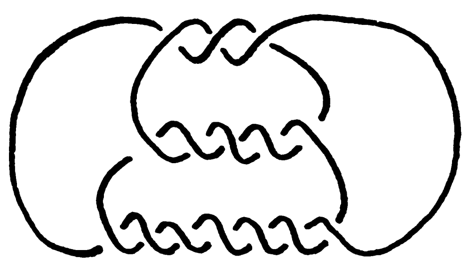

FIGURE 1

Lemma 3. *Suppose that* k *is a simple knot that is not reversible. Then* $X(k)$ *satisfies all the conditions for* X_1 *and* X_2 *in Lemma* 2.

Proof. As k is not reversible, $X(k)$ cannot be Seifert fibred, for the only knots with Seifert fibred exteriors are the (reversible) torus knots. Let λ and μ be a longitude and meridian for k. These are simple closed oriented curves in $\partial X(k)$ that represent generators of the kernels of the inclusion-induced maps $H_1(\partial X(k)) \to H_1(X(k))$ and $H_1(\partial X(k)) \to H_1(N(k))$; they represent a base for $H_1(\partial X(k))$ and are unique up to isotopy and orientation. Let $h\colon X(k) \to X(k)$ be an orientation preserving homeomorphism. It follows at once that $h(\lambda)$ is isotopic to $\pm\lambda$. However, the theorem of Gordon and Luecke [5], to the effect that knots are determined by their complements, implies that $h(\mu)$ is isotopic to $\pm\mu$. Thus, as h preserves orientation, it may be assumed after an isotopy that either $h(\lambda) = \lambda$ and $h(\mu) = \mu$ (in which case $h|\partial X(k)$ is isotopic to the identity), or $h(\lambda) = -\lambda$ and $h(\mu) = -\mu$. In the latter circumstance, h extends over $N(k)$ to produce an orientation preserving homeomorphism of S^3 that reverses the direction of k.

Note that for particular examples of knots it would often be possible to avoid the general but deep theorem from [5]. Lemmas 1, 2, and 3 now at once give a proof of the theorem: The required 3-manifolds can be formed by taking two distinct $K(p, q, r)$ pretzel knots, each with $|p|$, $|q|$, and $|r|$ distinct odd integers greater than 1, and identifying their boundaries together by means of homeomorphisms ϕ and $-\phi$.

The method described above exploits one way of obtaining links with the same Jones polynomials (namely, rotating a solid torus containing some components of the link); another well-known method uses Conway's idea of *mutation.* This has often been explained in recent times (see [13] for example). The standard example of two distinct knots related to one another by mutation is shown in Figure 2. The idea is that a ball, meeting one knot in two arcs, can be removed, given a rotation through angle π that permutes the four end points of the arcs, and then replaced to constitute the other knot. There are three possible axes for the rotation. For knot *diagrams* the idea becomes that of removing a disc meeting the diagram in two arcs, rotating the disc through π, or reflecting

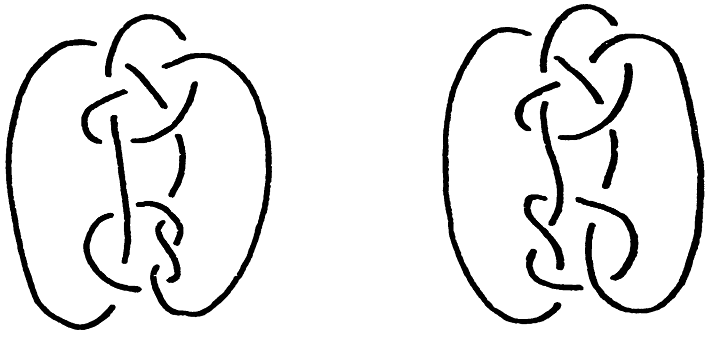

FIGURE 2

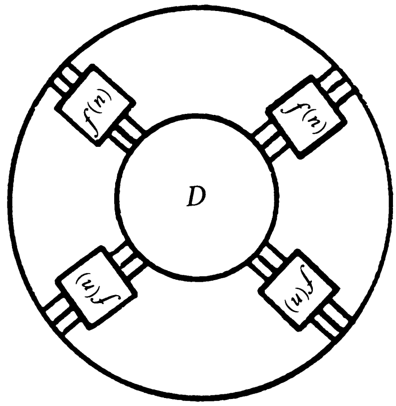

FIGURE 3

it and changing all the crossings, and then replacing it. The following result was first proved by Morton and Traczyk [16], but as a quick proof is now available in the spirit of [15] that is sketched here.

Lemma 4. *Suppose that D_1 and D_2 are knot diagrams that are related by mutation. Then $\Phi_{D_1} = \Phi_{D_2}$ as maps $\mathfrak{A} \to \mathbb{Z}[A, A^{-1}]$.*

Proof. It is sufficient to check that Φ_{D_1} and Φ_{D_2} agree on the base $\{S_n(\alpha)\}$ described in [15], where S_n is the nth Chebyshev polynomial (of the second kind). Now $S_n(\alpha)$ is the element of $\mathfrak{A}$ obtained by inserting into the annulus the idempotent $f^{(n)}$ of the nth Temperley-Lieb algebra and closing it off with n parallel arcs encircling the annulus. This $f^{(n)}$ has the property that $e_i f^{(n)} = 0 = f^{(n)} e_i$ for $1 \le i \le n-1$ where $1, e_1, e_2, \dots, e_{n-1}$ are the usual generators of the Temperley-Lieb algebra (see [14, 15]). As $f^{(n)} f^{(n)} = f^{(n)}$, the same element of $\mathfrak{A}$ results from using several (four, for example) copies of $f^{(n)}$ joined up in a chain around the annulus. Consider a disc D with four points on its boundary that are to be permuted by a mutation operation, and place n points on the boundary near each of these four. Consider the free module over $\mathbb{Z}[A, A^{-1}]$ generated by all link diagrams in the disc with these $4n$ end points, quotiented out by the relations (i) and (ii) previously described. The mutation operation induces a linear map on this module. The module has a base of all diagrams with no crossing and no closed curve. These diagrams are permuted nontrivially by the mutation. However, suppose that a copy of $f^{(n)}$ is placed in the plane just outside the disc D abutting it at each of the four n-tuples of points as shown in Figure 3. Then, because $e_i f^{(n)} = 0$, any base diagram containing an arc with both end points in the same n-tuple makes zero contribution to the Kauffman bracket polynomial of any link obtained by joining up the $4n$ points in the plane outside the whole ensemble of Figure 3. The remaining base diagrams are clearly invariant under mutation.

The same proof shows that if D_1 and D_2 are *link* diagrams that are related by a sequence of mutations, then

$$\Phi_{D_1}(S_n(\alpha), S_n(\alpha), \dots, S_n(\alpha)) = \Phi_{D_2}(S_n(\alpha), S_n(\alpha), \dots, S_n(\alpha)),$$

this being a slight extension to the above result.

Lemma 5. *Let k_1 and k_2 be oriented simple knots in S^3 related to each other by mutation (those of Figure 2, for example), and let k also be a simple oriented knot (for example that of Figure 1). Suppose that k, k_1, and k_2 are all distinct even when orientations are neglected. Let X_1, X_2, and X be their respective exteriors, and let M_1 and M_2 be the manifolds $X_1 \cup_\phi X$ and $X_2 \cup_\phi X$ where $\phi: S^1 \times S^1 \to S^1 \times S^1$ is any orientation reversing homeomorphism, the boundaries of X_1, X_2, and X being identified with $S^1 \times S^1$ by taking longitude and meridian to be standard generators of the torus. Then M_1 and M_2 are not homeomorphic but they have all the same $\mathrm{SU}(2)_q$ invariants.*

Proof. Torus knots have no nontrivial mutation (their double branched covers are atoroidal) so X_1 and X_2 are not Seifert fibred. As in the proof of Lemma 2, any homeomorphism from M_1 to M_2 would, after isotopy, have to send the incompressible torus in one to that of the other. That cannot happen as X_1, X_2, and X are the exteriors of distinct knots [5]. (Often the way to prove knots are distinct is to show their exteriors are different; the power of [5] is then not needed.)

Now, k_1 and k_2 have diagrams D_1 and D_2 that are related by mutation. As in the proof of Lemma 1, X can be obtained by surgery on a solid torus, using a framed link represented by a diagram Δ in an annulus, in such a way that insertion of Δ around D_1 gives a surgery diagram E_1 for M_1, insertion around D_2 gives a diagram E_2 for M_2. Then $\Phi_{E_1}(\mathbf{a}, \mathbf{a}, \dots, \mathbf{a}) = \Phi_{D_1}(\Psi_\Delta(\mathbf{a}, \mathbf{a}, \dots, \mathbf{a}))$, but by Lemma 4 this is $\Phi_{D_2}(\Psi_\Delta(\mathbf{a}, \mathbf{a}, \dots, \mathbf{a}))$, which in turn equals $\Phi_{E_2}(\mathbf{a}, \mathbf{a}, \dots, \mathbf{a})$. However E_1 and E_2 have the same linking matrix and hence the manifolds they represent (by means of surgery) have the same $\mathrm{SU}(2)_q$ invariants.

This lemma provides a second proof of the theorem.

References

1. C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel, *Three-manifold invariants derived from the Kauffman bracket*, preprint, Université de Nantes 1991.
2. F. Bonahon and L. C. Siebenmann, *New geometric splittings of classical knots*, London Math. Soc. Lecture Note Ser., vol. 75, Cambridge Univ. Press, Cambridge and New York.
3. G. Burde and H. Zieschang, *Knots*, De Gruyter, Berlin and New York, 1985.
4. J. H. Conway, *An enumeration of knots and links*, Computational problems in abstract algebra (J. Leech, ed.), Pergamon Press, New York, 1969, pp. 329–358.
5. C. McA. Gordon and J. Luecke, *Knots are determined by their complements*, J. Amer. Math. Soc. **2** (1989), 371–415.
6. J. Hempel, *Construction of orientable* 3*-manifolds*, Topology of 3-manifolds and related topics (M. K. Fort Jr., ed.), Prentice Hall, Englewood Cliffs, NJ, 1962, pp. 207–212.
7. ——, 3*-manifolds*, Ann. of Math. Stud., vol. 86, Princeton Univ. Press, Princeton, NJ, 1976.
8. W. H. Jaco and P. B. Shalen, *Seifert fibered spaces in* 3*-manifolds*, Mem. Amer. Math. Soc., vol. 21, no. 220, Amer. Math. Soc., Providence, RI, 1979.
9. K. Johannson, *Homotopy equivalences of* 3*-manifolds with boundary*, Lecture Notes in Math., vol. 761, Springer-Verlag, Berlin, Heidelberg, New York, 1979.
10. R. C. Kirby and P. Melvin, *The* 3*-manifold invariants of Witten and Reshetikhin-Turaev for* $\mathbf{sl}(2, \mathbb{C})$, Invent. Math. **105** (1991), 473–545.

11. W. B. R. Lickorish, *Linear skein theory and link polynomials*, Topology Appl. **27** (1987), 265–274.

12. ——, *The panorama of polynomials for knots, links and skeins*, Braids (J. S. Birman and A. Libgober, eds.), Contemp. Math., vol. 78, Amer. Math. Soc., Providence, RI, 1988, pp. 399–414.

13. ——, *Polynomials for links*, Bull. London Math. Soc. **20** (1988), 558–588.

14. ——, *Three-manifolds and the Temperley-Lieb algebra*, Math. Ann. **290** (1991), 657–670.

15. ——, *Calculations with the Temperley-Lieb algebra*, Preprint, Univ. of Cambridge, 1990.

16. H. R. Morton and P. Traczyk, *The Jones polynomials of satellite links around mutants*, Braids (J. S. Birman and A. Libgober, eds.), Contemp. Math., vol. 78, Amer. Math. Soc. Providence, RI, 1988, pp. 575–585.

17. N. Y. Reshetikhin and V. G. Turaev, *Invariants of* 3*-manifolds via link polynomials and quantum groups*. Invent. Math. **103** (1991), 547–597.

18. H. F. Trotter, *Non-invertible knots exist*, Topology **2** (1964), 341–358.

Department of Pure Mathematics and Mathematical Statistics, The University of Cambridge, 16 Mill Lane, Cambridge, CB2 1SB, England

E-mail address: wbrl@PHX.CAM.AC.UK