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WHEN IS $F[x, y]$ A UNIQUE FACTORIZATION DOMAIN?

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ABSTRACT. Although the commutative polynomial ring $F[x, y]$ is a unique factorization domain (UFD) and the free associative algebra $F\langle x, y\rangle$ is a similarity-UFD when F is a (commutative) field, it is shown that the polynomial ring $F[x, y]$ in two commuting indeterminates is not a UFD in any reasonable sense when F is the skew field of rational quaternions.

As anyone who has had a course in abstract algebra knows, the polynomial ring $F[x, y]$ in two variables over a field F is a unique factorization domain (UFD). In generalizing to the noncommutative case there are at least two natural possibilities to consider.

First we take x and y to be noncommutative while the field of coefficients remains commutative. Specifically, we consider the free associative algebra $R = F\langle x, y\rangle$. It can be shown that this ring is a similarity-UFD. Thus for two factorizations

$$p_1 p_2 \cdots p_n = q_1 q_2 \cdots q_m,$$

where the p_i and q_j are irreducibles in R, $n = m$, and there is a permutation σ of the subscripts such that p_i and $q_{\sigma(i)}$ are similar, which means $R/p_iR \cong R/q_{\sigma(i)}R$ as R-modules (see [1, p. 9]). For the second case we assume that x and y commute but take F to be a skew field. It is our purpose to show that in this situation factorization into irreducibles is not generally unique in any reasonable sense.

If R is any (not necessarily commutative) integral domain then an atom (or irreducible) in R is a nonzero nonunit that has no proper factors. Recall (see [2, p. 159]) that elements a and a' are similar (i.e., $R/aR \cong R/a'R$ as R-modules) if and only if there exists $b \in R$ such that

$$aR + bR = R, \qquad aR \cap bR = ba'R.$$

If, in addition, a is a central atom then $ba \in aR \cap bR = ba'R$ so that $aR \subseteq a'R \subset R$, which implies $aR = a'R$. This establishes the following result.

Proposition 1. *Let R be any (not necessarily commutative) integral domain with similar elements a and a'. If a is a central atom then a and a' are (right) associates.*

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In the remainder of this note we shall let $R = F[x, y]$ where $F = \mathbb{Q}(1, i, j, k)$ is the quaternion algebra over the field $\mathbb{Q}$ of rational numbers. We refer to [4] for any needed results on quaternions. Thus F is a skew field in which each element a has the form

$$a = a_0 + a_1 i + a_2 j + a_3 k$$

where a_i is in the center $C(F)$ of F, which is the field $\mathbb{Q}$. Similarly, each $f \in R$ has the form

$$f = f_0 + f_1 i + f_2 j + f_3 k$$

where f_i is in the center $C(R)$ of R, which is $\mathbb{Q}[x, y]$. The conjugate of f is defined to be

$$\bar{f} = f_0 - f_1 i - f_2 j - f_3 k.$$

Note that

$$f\bar{f} = f_0^2 + f_1^2 + f_2^2 + f_3^2 \in \mathbb{Q}[x, y].$$

Furthermore, conjugation is an antiautomorphism of $R = F[x, y]$. We refer to irreducible polynomials as atoms and to nonconstant central polynomials that have no proper central factors as C-atoms.

In the polynomial ring $F[x]$ the relationship between atoms and C-atoms is easily determined. We shall describe the situation for the polynomial ring $K[x]$ where K is the quaternion algebra over any field of characteristic $\neq 2$ and over which the equation

$$a^2 + b^2 + c^2 + d^2 = 0$$

has only the trivial solution (these two conditions ensure that K is a noncommutative field [4, p. 301]). Conjugation is defined and is an antiautomorphism of $K[x]$, analogous with the above description. Recall that $K[x]$ is a similarity-UFD [1].

Proposition 2. *Let K be the quaternion algebra just described. Let f be a polynomial in $K[x]$ that is not associated to a central polynomial. Then f is an atom if and only if $f\bar{f}$ is a C-atom.*

Proof. Assume that f is an atom. Since $f \to \bar{f}$ is an antiautomorphism of $K[x]$, $\bar{f}$ is also an atom. Suppose $f\bar{f} = gh$ where g and h are central nonunits. Unique factorization in $K[x]$ shows that both g and h are atoms and f is similar to g or to h. Proposition 1 then shows that f is associated to g or to h and this contradicts the hypothesis. Conversely, if $f\bar{f}$ is a C-atom and $f = rs$ where r, $s \in K[x]$ then $f\bar{f} = r\bar{r}s\bar{s}$, which forces either $r\bar{r}$ or $s\bar{s}$ to be a unit. Thus either r or s is a unit and f is an atom.

We shall now show that Proposition 2 is not valid for polynomials in two variables. Consider the polynomial ring $R = F[x, y]$ over the rational quaternions and the polynomial

$$f = (x^2y^2 - 1) + (x^2 - y^2)i + 2xyj.$$

Then $f\bar{f}$ is not a C-atom; we have

$$f\bar{f} = (x^2y^2 - 1)^2 + (x^2 - y^2)^2 + 4x^2y^2 = (x^4 + 1)(y^4 + 1), \tag{1}$$

which is a product of two C-atoms. Note that the unique factorization

$$x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$$

that is carried out over the reals shows that the first factor in (1) is a C-atom in R (i.e., irreducible over the rationals).

We claim that f is an atom. If we write

$$f = ax^2 + bx + c$$

where

$$a = y^2 + i, \qquad b = 2yj, \qquad c = -1 - y^2 i,$$

and if we view f in $F(y)[x]$, then $f\bar{f}$ is a C-atom [3, p. 339], and so f is an atom there by Proposition 2. Thus the only possible proper factorization of f in $F[x, y]$ is of the form

$$f = r(y)s(x) \quad (\text{or } f = s(x)r(y)).$$

But then $r(y)$ must be a common left (or right) factor of a, b, and c, and so $r(y)\bar{r}(y)$ must be a common factor of

$$a\bar{a} = c\bar{c} = y^4 + 1 \quad \text{and} \quad b\bar{b} = 4y^2.$$

Since this is possible only if $r(y)\bar{r}(y)$ is a unit, we conclude that f can have no proper factorization; that is, f is an atom. Thus Proposition 2 fails in this case.

Equation (1) may be written

$$f\bar{f} = (x^2 + i)(x^2 - i)(y^2 + i)(y^2 - i), \tag{2}$$

showing a product of two atoms equal to a product of four atoms (the latter are atoms by Proposition 2). Atomic factorization in R is not unique.

We shall show that the behavior of degree 1 atoms in R is more predictable than that exhibited in (2).

Proposition 3. *Let f be a linear polynomial in $R = F[x, y]$ that is not associated to a central polynomial. Then $f\bar{f}$ is a C-atom.*

Proof. Let $f = ax + by + c$. If either a or b is 0 then $f \in F[y]$ or $F[x]$, respectively, so $f\bar{f}$ is a C-atom by Proposition 2. Suppose now that both a and b are nonzero and $f\bar{f} = rs$ where $r, s \in C(R)$. Viewing this equation in $F(x)[y]$ we see that $f\bar{f}$ must be a C-atom in $F(x)[y]$ so that either r or s is a unit, that is, a member of $F(x)$. Let us assume (without loss in generality) that it is r, so that $r \in F[x]$. Similarly we find that r or s is a member of $F(y)$. If r is a unit in $F(y)$ then $r \in F[y] \cap F[x] = F$ and we are finished. If s is a unit in $F(y)$ then $s \in F[y]$. However, the equation $f\bar{f} = r(x)s(y)$ is not possible when $a \neq 0$ and $b \neq 0$. Thus $f\bar{f}$ is a C-atom in R.

Corollary. *Any product of linear polynomials in $R = F[x, y]$ is unique in the sense that if*

$$p_1 p_2 \cdots p_n = q_1 q_2 \cdots q_m$$

where the p_i, q_j are degree one polynomials then $n = m$ and there is a permutation σ of the subscripts such that $p_i\bar{p}_i$ are $q_{\sigma(i)}\bar{q}_{\sigma(i)}$ are associates. If p_i is associated to a central polynomial then p_i and $q_{\sigma(i)}$ are associates.

Proof. The result follows from the equation

$$p_1\bar{p}_1 p_2\bar{p}_2 \cdots p_n\bar{p}_n = q_1\bar{q}_1 q_2\bar{q}_2 \cdots q_m\bar{q}_m$$

of C-atoms in the unique factorization domain $\mathbb{Q}[x, y]$. Note that if p_i is associated to a C-atom then $p_i\bar{p}_i$ is associated to the square of that C-atom; otherwise, $p_i\bar{p}_i$ is itself a C-atom.

It is difficult to improve on the corollary: (2) shows a product of two atoms equal to a product of four atoms. Thus when F is the skew field of rational quaternions, $R = F[x, y]$ cannot be considered to be a unique factorization domain in any reasonable sense.

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