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ON THE NONAUTONOMOUS VOLTERRA-LOTKA COMPETITION EQUATIONS

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ABSTRACT. A nonautonomous competitive Lotka-Volterra system of two equations is considered. It is shown that if the coefficients are continuous and satisfy certain inequalities, then any solution that is positive at some point has the property that one of its components vanishes while the other approaches a certain solution of the logistic equation.

1. INTRODUCTION

Consider the nonautonomous system of differential equations

$$(*)\qquad \begin{cases} u'(t) = u(t)[a(t) - b(t)u(t) - c(t)v(t)], \\ v'(t) = v(t)[d(t) - e(t)u(t) - f(t)v(t)], \end{cases}$$

where the functions $a(t), b(t), \dots,$ and $f(t)$ are assumed to be continuous and bounded above and below by positive constants. Given a function $g(t)$, we let g_L and g_M denote $\inf_{-\infty<t<\infty} g(t)$ and $\sup_{-\infty<t<\infty} g(t)$, respectively. In [2] it was shown that if the inequalities

$$a_L f_L > c_M d_M \quad \text{and} \quad b_L d_L > a_M e_M$$

hold, and if the coefficients $a(t), \dots,$ and $f(t)$ are almost periodic, then $(*)$ has a unique almost periodic solution whose components are bounded above and below by positive constants, which is globally asymptotically stable. The general case, not involving almost periodicity, was considered in [1]. The periodic case was studied in [4]. A similar system, involving reaction-diffusion equations, with periodic coefficients was studied in [3]. For the ecological significance of the system $(*)$, the reader is referred to [7] and the works cited above.

The purpose of this paper is to study $(*)$ under the assumption that the inequalities

$$(1)\qquad a_L f_L > c_M d_M \quad \text{and} \quad b_M d_M \le a_L e_L$$

hold.

In particular, we will show that if $\mathrm{col}(u(t), v(t))$ is a solution of $(*)$ satisfying the inequalities $u(t_0) > 0$ and $v(t_0) > 0$ for some number t_0, then

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$\lim_{t\to\infty} v(t) = 0$ and $\lim_{t\to\infty}[u(t) - u^*(t)] = 0$, where $u^*(t)$ is the unique solution of the logistic equation

$$u'(t) = u(t)[a(t) - b(t)u(t)] \tag{2}$$

such that $0 < \delta \le u^*(t) \le \Delta < \infty$ for certain numbers δ and Δ.

Of course, a similar result, where the roles of u and v are interchanged, will hold if the inequalities in (1) are replaced by

$$a_M f_M \le c_L d_L \quad \text{and} \quad b_L d_L > a_M e_M .$$

This generalizes the constant coefficient case, which can be easily verified from a phase space analysis of the system (see, e.g., [7]).

We wish to point out that the main result in this paper and some of the methods used have been motived by [3]. Also, Zhou and Pao [8] established a somewhat similar result for a system of reaction-diffusion equations where the coefficients were assumed to be positive constants. Also see Gopalsamy [6].

2. Preliminary lemmas

Henceforth we shall assume that the functions $a(t)$, $b(t)$, ..., and $f(t)$ are continuous, bounded above and below by positive constants, and satisfy the inequalities (1).

Lemma 1. *Let* $\mathrm{col}(u(t), v(t))$ *be a solution of* $(*)$ *such that* $u(t_0) > 0$ *and* $v(t_0) > 0$. *If* $u(t) \ge \varepsilon$ *for all* $t \ge t_0$, *where* ε *is a positive number, then* $\lim_{t\to\infty} v(t) = 0$.

Proof. It follows (see [1]) that $u(t)$ and $v(t)$ are bounded and positive for all t, $t \ge t_0$. Let $\underline{u} = \lim_{t\to\infty} \inf u(t)$ and $\overline{v} = \lim_{t\to\infty} \sup v(t)$. Then $\underline{u} \ge \varepsilon > 0$ and $\overline{v} \ge 0$. It suffices to show that $\overline{v} = 0$. Suppose that $\overline{v} > 0$. In order to obtain a contradiction, we first establish the inequality

$$a_L \le b_M \underline{u} + c_M \overline{v} . \tag{3}$$

We consider the following two cases:

Case 1. Suppose that $u'(t)$ has arbitrarily large zeros. Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of zeros of $u'(t)$, where $s_n \to \infty$ as $n \to \infty$. Let $\{\tau_n\}_{n=1}^{\infty}$ be a sequence of numbers such that $\tau_n \to \infty$ and $u(\tau_n) \to \underline{u}$ as $n \to \infty$. We can assume that for each n there exists an integer m_n satisfying $s_{m_n} < \tau_n < s_{m_{n+1}}$. Let $\sigma_n \in [s_{m_n}, s_{m_{n+1}}]$ such that $u(\sigma_n)$ is the minimum of $u(t)$ on the interval $[s_{m_n}, s_{m_{n+1}}]$. Since $u(\sigma_n) \le u(\tau_n)$, it follows that $\limsup u(\sigma_n) \le \liminf u(\sigma_n)$. Hence, $\lim_{n\to\infty} u(\sigma_n)$ exists and $\lim_{n\to\infty} u(\sigma_n) = \underline{u}$. Clearly, $u'(\sigma_n) = 0$. Therefore, from the first equation of $(*)$, we obtain $a(\sigma_n) = b(\sigma_n)u(\sigma_n) + c(\sigma_n)v(\sigma_n)$ and hence $a_L \le b_M u(\sigma_n) + c_M \sup_{t\ge\sigma_n} v(t)$. By taking the limit of the right-hand side as $n \to \infty$, we obtain the inequality $a_L \le b_M \underline{u} + c_M \overline{v}$.

Case 2. Now suppose that $u'(t) \ne 0$ for $t \ge t_1$ for some number t_1. Then $\lim_{t\to\infty} u(t)$ exists and $\lim_{t\to\infty} u(t) = \underline{u}$. Since $u(t)$ is bounded, there exists a sequence $\{\zeta_n\}_{n=1}^{\infty}$ such that $u'(\zeta_n) \to 0$ as $n \to \infty$. Hence, $a(\zeta_n) - b(\zeta_n)u(\zeta_n) - c(\zeta_n)v(\zeta_n) \to 0$ as $n \to \infty$. Since $a_L - b_M u(\zeta_n) - c_M \sup_{t\ge\zeta_n} v(t) \le a(\zeta_n) - b(\zeta_n)u(\zeta_n) - c(\zeta_n)v(\zeta_n)$, we obtain, by taking limits, $a_L - b_M\underline{u} - c_M\overline{v} \le 0$.

It follows from a similar argument that

$$d_M \ge e_L \underline{u} + f_L \overline{v} . \tag{4}$$

For example, in Case 1, where $v'(t)$ would have arbitrarily large zeros, one would let $v(\sigma_n)$ to be the maximum of $v(t)$ on $[s_{m_n}, s_{m_{n+1}}]$.

Now, multiplying (4) by $-c_M/f_L$ and adding the result to (3), we obtain

$$a_L - \frac{c_M}{f_L} d_M \le \left[b_M - \frac{c_M}{f_L} e_L \right] \underline{u}.$$

Thus it follows from (1) and the fact that $u \ge \varepsilon > 0$ that $(b_M f_L - c_M e_L)/f_L > 0$. Similarly, multiplying (3) by $-e_L/b_M$ and adding it to (4), we obtain $[(b_M f_L - c_M e_L)/b_M]\overline{v} \le 0$. Since from the above inequality we have $b_M f_L - c_M e_L > 0$ and $b_M > 0$, it follows that $\overline{v} \le 0$, which is a contradiction. This completes the proof of the lemma.

Lemma 2. *Let k and ε be numbers such that $k > d_M/f_L$, $\varepsilon > 0$, and $a_L - b_M\varepsilon - c_M k > 0$. If $\mathrm{col}(\hat{u}(t), \hat{v}(t))$ is a solution of $(*)$ such that $\hat{u}(t_0) = \varepsilon$ and $\hat{v}(t_0) = k$, then $\hat{v}(t) \to 0$ as $t \to \infty$.*

Proof. We note that

$$\begin{aligned}\hat{u}'(t_0) &= \hat{u}(t_0)[a(t_0) - b(t_0)\hat{u}(t_0) - c(t_0)\hat{v}(t_0)] \\ &\ge \varepsilon[a_L - b_M\varepsilon - c_M k] > 0,\end{aligned}$$

$$\begin{aligned}\hat{v}'(t_0) &= \hat{v}(t_0)[d(t_0) - e(t_0)\hat{u}(t_0) - f(t_0)\hat{v}(t_0)] \\ &\le k[d_M - f(t_0)k] < 0\end{aligned}$$

since $k > d_M/f_L$. We wish to show that $\hat{u}(t) > \varepsilon$ and $\hat{v}(t) < k$ for all $t > t_0$. These inequalities certainly hold for t close to t_0 and $t > t_0$ since $\hat{u}'(t_0) > 0$ and $\hat{v}'(t_0) < 0$. If they did not hold for all $t > t_0$, then there would exist a number $\bar{t}$ such that $\hat{u}(t) > \varepsilon$ and $\hat{v}(t) < k$ for $t_0 < t < \bar{t}$ and either (a) $\hat{u}(\bar{t}) = \varepsilon$ or (b) $\hat{v}(\bar{t}) = k$. If (a) held, then we would have $\hat{u}(\bar{t}) \le 0$ and $\hat{v}(\bar{t}) \le k$. Therefore, we would have

$$0 \ge \hat{u}'(\bar{t}) = \hat{u}(\bar{t})[a(\bar{t}) - b(\bar{t})\hat{u}(\bar{t}) - c(\bar{t})\hat{v}(\bar{t})] \ge \varepsilon[a_L - b_M\varepsilon - c_M k] > 0,$$

a contradiction. If (b) held, then we would have $0 \le \hat{v}'(\bar{t})$ and $\varepsilon \le \hat{u}(\bar{t})$. But $\hat{v}'(\bar{t}) = k[d(\bar{t}) - e(\bar{t})\hat{u}(\bar{t}) - f(\bar{t})k] < k[d_M - f_L k] < 0$, again a contradiction. The assertion of this lemma now follows from Lemma 1.

Lemma 3. *There exists a unique solution $u^*(t)$ of the logistic equation*

$$u'(t) = u(t)[a(t) - b(t)u(t)] \tag{L}$$

such that $\delta \le u^*(t) \le \Delta$ on $(-\infty, \infty)$, where Δ and δ are any numbers satisfying the inequalities $0 < \delta < a_L/b_M$ and $a_M/b_L < \Delta$.

It appears that this lemma ought to be known. In fact, the referee of this paper has pointed out that it follows from [5]. While this appears to be the case, it is not obvious. Therefore, for the convenience of the reader we give here an independent and elementary proof.

Proof. For each positive integer n, let $u_n(t)$ be the solution of (L) satisfying $u_n(-n) = \Delta$. Then $u_n'(-n) = \Delta[a(-n) - b(-n)\Delta] \le \Delta[a_M - b_L\Delta] < 0$. Hence for $t + n$ small and positive we have $\delta < u_n(t) < \Delta$. It follows that this inequality holds for all $t > -n$. For, suppose not. Then there exists a number $\bar{t} > -n$, such that $\delta < u_n(t) < \Delta$ for $-n < t < \bar{t}$ and either (i) $u_n(\bar{t}) = \delta$ or (ii) $u_n(\bar{t}) =$

Δ. In the first case we must have $u_n'(\bar{t}) \leq 0$. But $u_n'(\bar{t}) = \delta[a(\bar{t}) - b(\bar{t})\delta] > \delta[a_L - b_M\delta] > 0$, a contradiction. In the second case we have $u_n'(\bar{t}) \geq 0$. But $u_n'(\bar{t}) = \Delta[a(\bar{t}) - b(\bar{t})\Delta] \leq \Delta[a_M - b_L\Delta] < 0$, again a contradiction. This shows that $\delta < u_n(t) < \Delta$ holds for all $t > -n$. In particular, $\delta < u_n(0) < \Delta$ holds for all positive integers n. Therefore, there exists a subsequence $\{u_{n_k}(0)\}_{k=1}^{\infty}$ of $\{u_n(0)\}$ such that $u_{n_k}(0) \to u_0$ as $k \to \infty$, $\delta \leq u_0 \leq \Delta$. Let $u^*(t)$ be the solution of (L) such that $u^*(0) = u_0$. Then, since each $u_{n_k}(t)$ satisfies (L) and $u_{n_k}(0) \to u_0$ as $k \to \infty$, it follows that $u_{n_k}(t) \to u^*(t)$ uniformly with respect to t on compact subintervals of $(-\infty, \infty)$. Since for each number t_1, $\delta < u_{n_k}(t_1) < \Delta$ if $-n_k < t_1$, we must have $\delta \leq u^*(t_1) \leq \Delta$.

In order to establish the uniqueness, we assume that (L) has two solutions U_1 and U_2 satisfying $\delta \leq U_k(t) \leq \Delta$ for $k = 1, 2$ and $t \in (-\infty, \infty)$. Since we have a first-order differential equation, we can assume, by uniqueness, that $\delta < U_1(t) < U_2(t) < \Delta$. Now, $(d/dt)\ln U_1 - (d/dt)\ln U_2 = b(t)(U_2(t) - U_1(t)) > 0$. This shows that $\ln U_1(t)/U_2(t)$, and hence $U_1(t)/U_2(t)$ is strictly increasing. Thus, $U_1(t)/U_2(t) < U_1(0)/U_2(0) < 1$ for $t < 0$ and

$$\begin{aligned}\frac{d}{dt}\ln\frac{U_1(t)}{U_2(t)} &= b(t)[U_2(t) - U_1(t)] \geq b_L U_2(t)\left[1 - \frac{U_1(t)}{U_2(t)}\right] \\ &\geq b_L\delta\left[1 - \frac{U_1(0)}{U_2(0)}\right] > 0 \quad \text{for } t \leq 0.\end{aligned}$$

Integrating from T to 0, $T < 0$, we obtain

$$\begin{aligned}\ln\frac{U_1(0)}{U_2(0)} - \ln\frac{U_1(T)}{U_2(T)} &= \int_T^0 b(t)(U_2(t) - U_1(t))\,dt \\ &\geq \int_T^0 b_L\delta\left[1 - \frac{U_1(0)}{U_2(0)}\right] dt > 0.\end{aligned}$$

Therefore,

$$\ln\frac{U_1(T)}{U_2(T)} \leq \ln\frac{U_1(0)}{U_2(0)} - \int_T^0 b_L\delta\left[1 - \frac{U_1(0)}{U_2(0)}\right] dt.$$

This shows that $\lim_{T\to-\infty}\ln U_1(T)/U_2(T) = -\infty$. Consequently, since $\delta \leq U_1(t) < U_2(t) < \Delta$, we conclude that $U_1(T)/U_2(T) \to 0$ as $T \to -\infty$. But, $U_1(T)/U_2(T) \geq \delta/U_2(T) \geq \delta/\Delta$, which leads to a contradiction. This completes the proof of Lemma 3.

Lemma 4. *Let k, ε, and δ be numbers such that $k > d_M/f_L$, $0 < \varepsilon < \delta < a_L/b_M$, and $a_L - b_M\varepsilon - c_M k > 0$. If $\mathrm{col}(\hat{u}(t), \hat{v}(t))$ is a solution of $(*)$ such that $\mathrm{col}(\hat{u}(t_0), \hat{v}(t_0)) = \mathrm{col}(\varepsilon, k)$, then $u^*(t) - \hat{u}(t) \to 0$ as $t \to \infty$, where $u^*(t)$ is the unique solution of Lemma* 3.

Proof. As shown in the proof of Lemma 2, $\hat{u}(t) \geq \varepsilon$ for $t \geq t_0$. Moreover, (see [1]), $\hat{u}$ is bounded above for $t \geq t_0$. Let $w(t) = 1/\hat{u}(t)$ and $w^*(t) = 1/u^*(t)$ for $t \geq t_0$. We have

$$\begin{aligned}w'(t) &= -a(t)w(t) + b(t) + c(t)\hat{v}(t)w(t), \\ w^{*'}(t) &= -a(t)w^*(t) + b(t),\end{aligned}$$

and hence

$$w'(t) - w^{*'}(t) = -a(t)(w(t) - w^*(t)) + c(t)\hat{v}(t)w(t) \tag{5}$$

for $t \geq t_0$.

We consider two possibilities:

(I) There exists $t_1 \geq t_0$ such that $(w - w^*)'(t) \neq 0$ for $t \geq t_1$.

(II) There exists a sequence of numbers $\{s_n\}_1^\infty$ in $[t_0, \infty)$ such that for $n \geq 1$, $s_n < s_{n+1}$, $(w - w^*)'(s_n) = 0$, and $s_n \to \infty$ as $n \to \infty$.

If (I) holds, then $\lim_{t\to\infty}(w(t) - w^*(t))$ exists. If $\lim_{t\to\infty}(w(t) - w^*(t)) = 0$, then, since $\hat{u}(t)$ and $u^*(t)$ are bounded and

$$u^*(t) - \hat{u}(t) = u^*(t)\hat{u}(t)(w(t) - w^*(t)),$$

it follows that $u^*(t) - \hat{u}(t) \to 0$ as $t \to \infty$. If (I) holds and $\lim_{t\to\infty}(w(t) - w^*(t)) \neq 0$, then since $a(t) \geq a_L > 0$ and, according to Lemma 2, $\hat{v}(t) \to 0$ as $t \to \infty$, (5) implies the existence of numbers $\alpha > 0$ and $t_2 \geq t_1$ such that $|(w - w^*)'(t)| \geq \alpha$ for all $t \geq t_2$. Since this contradicts the boundedness of $w(t) - w^*(t)$ on $[t_0, \infty)$, it follows that if (I) holds, then $\lim_{t\to\infty}(u(t) - u^*(t)) = 0$.

If (II) holds, let $\tau_n \in [s_n, s_{n+1}]$ be chosen for each $n \geq 1$ such that

$$|w(\tau_n) - w^*(\tau_n)| = \max_{s_n \leq t \leq s_{n+1}} |w(t) - w^*(t)|. \tag{6}$$

Since $(w - w^*)'(s_n) = 0$ for $n \geq 1$, it follows that $(w - w^*)'(\tau_n) = 0$ for $n \geq 1$. Therefore, by (5), $w(\tau_n) - w^*(\tau_n) = c(\tau_n)\hat{v}(\tau_n)w(\tau_n)/a(\tau_n)$.

Since $a(t) \geq a_L$, $w(t)$ and $c(t)$ are bounded, and $\hat{v}(t) \to 0$ as $t \to \infty$, we see that

$$\lim_{n\to\infty}(w(\tau_n) - w^*(\tau_n)) = 0. \tag{7}$$

Since $s_n \to \infty$ as $n \to \infty$, it follows from (6) and (7) that $w(t) - w^*(t) \to 0$ as $t \to \infty$. Therefore, if (II) holds we have $\lim_{t\to\infty}(u^*(t) - u(t)) = 0$.

Since the possibilities (I) and (II) are exhaustive, the lemma is proved.

Lemma 5. *Let $k_1 > \Delta$, where Δ is a number as in Lemma 3. If $\tilde{u}(t)$ is a solution of* (L) *satisfying $\tilde{u}(t_0) = k_1$, then $\tilde{u}(t) - u^*(t) \to 0$ as $t \to \infty$.*

Proof. Since $\tilde{u}(t_0) > u^*(t_0)$, it follows that $\tilde{u}(t) > u^*(t)$ for all t in $(-\infty, \infty)$. Let $w^*(t) = 1/u^*(t)$ and $\tilde{w}(t) = 1/\tilde{u}(t)$. Then, $w^{*\prime} = -aw^* + b$ and $\tilde{w}' = -a\tilde{w} + b$. Hence,

$$\tilde{w}(t) - w^*(t) = e^{-\int_{t_0}^t a(s)\,ds}(\tilde{w}(t_0) - w^*(t_0)) \quad \text{for } t \geq t_0.$$

But,

$$-\int_{t_0}^t a(s)\,ds \leq -a_L(t - t_0), \qquad t \geq t_0.$$

Therefore, $\tilde{w}(t) - w^*(t) \to 0$ as $t \to \infty$, and hence $(u^*(t) - \tilde{u}(t))/u^*(t)\tilde{u}(t) \to 0$ as $t \to \infty$. Since $\tilde{u}(t) > u^*(t) \geq \delta$, we conclude that $\tilde{u}(t) - u^*(t) \to 0$ as $t \to \infty$.

Lemma 6. *Let k and k_1 be numbers as defined earlier. If* $\mathrm{col}(u(t), v(t))$ *is a solution of $(*)$ such that $0 < u(t_0) < k_1$ and $0 < v(t_0) < k$, then $u(t) - u^*(t) \to 0$ and $v(t) \to 0$ as $t \to \infty$.*

Proof. We may assume that ε in Lemma 2 satisfies the inequalities $0 < \varepsilon < a_M/b_L < \Delta < k_1$, $\varepsilon < u(t_0)$, and $a_L - b_M\varepsilon - c_M k > 0$. Recall that $\mathrm{col}(\hat{u}(t), \hat{v}(t))$ is the solution of $(*)$ such that $\hat{u}(t_0) = \varepsilon$ and $\hat{v}(t_0) = k$. We note that

$\operatorname{col}(\tilde{u}(t), 0)$ is also a solution of $(*)$, where $\tilde{u}(t)$ is the solution of Lemma 5 satisfying the initial condition $\tilde{u}(t_0) = k_1$. Since $\hat{u}(t_0) < u(t_0) < \tilde{u}(t_0)$ and $\hat{v}(t_0) > v(t_0) > \tilde{v}(t_0)$, we have $\hat{u}(t) < u(t) < \tilde{u}(t)$ and $\hat{v}(t) > v(t) > \tilde{v}(t)$ for all $t \geq t_0$ (see [1]), where $\tilde{v}(t)$ denotes the second component of the solution $\operatorname{col}(\tilde{u}(t), 0)$. Since $\tilde{v}(t) \equiv 0$, and $\hat{v}(t) \to 0$ as $t \to \infty$, it follows that $v(t) \to 0$ as $t \to \infty$. Similarly, since $\hat{u}(t) - u^*(t) < u(t) - u^*(t) < \tilde{u}(t) - u^*(t)$, and since $\hat{u}(t) - u^*(t) \to 0$ and $\tilde{u}(t) - u^*(t) \to 0$ as $t \to \infty$, we obtain the desired result that $u(t) - u^*(t) \to 0$ as $t \to \infty$.

3. Main result

We are now ready to prove our main result.

Theorem. *Assume that $a(t), b(t), \ldots,$ and $f(t)$ are continuous, bounded above and below by positive constants, and satisfy the inequalities in* (1). *If $\operatorname{col}(u(t), v(t))$ is any solution of $(*)$ such that $u(t_0) > 0$ and $v(t_0) > 0$ for some t_0 in $(-\infty, \infty)$, then $v(t) \to 0$ and $u(t) - u^*(t) \to 0$ as $t \to \infty$, where $u^*(t)$ is the solution of the logistic equation described in Lemma* 3.

Proof. In view of the above lemmas, it suffices to show that there exists a number $t_1 \geq t_0$, such that $0 < u(t_1) < k_1$, and $0 < v(t_1) < k$. To this end, suppose that $u(t) \geq k_1$ for $t \geq t_0$ (recall that $k_1 > \Delta > a_M/b_L$). Then $u'(t) = u(t)[a(t) - b(t)u(t) - c(t)v(t)] \leq u(t)[a(t) - b(t)k_1] < 0$. Hence, $u'(t)/u(t) \leq a(t) - b(t)k_1 < a_M - b_L k_1 < 0$. But this implies that $\ln u(t) \to -\infty$, and hence $u(t) \to 0$ as $t \to \infty$, which is a contradiction. This shows that there exists a number $\bar{t}_1 \geq t_0$ such that $u(\bar{t}_1) < k_1$. Similarly, there exists a number $\bar{t}_2 \geq t_0$, such that $v(\bar{t}_2) < k$. Let $t_1 = \max(\bar{t}_1, \bar{t}_2)$, and the proof is complete.

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