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DIFFERENTIATION OF ZYGMUND FUNCTIONS

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ABSTRACT. The "little-o Zygmund class" λ^* contains a nowhere-differentiable function.

0. INTRODUCTION

A classical result due originally to Rajchman and then improved by Zygmund [ZY, p. 43] states that if $f \in \lambda^*(T)$ and f is real valued then f must be differentiable on a dense subset of T. This implies that if $F \in \mathfrak{B}_o$ (the "little-o Bloch space") then $\mathrm{Re}(F)$ must possess a radial (and hence nontangential) limit at each point of a dense subset of the boundary.

Somewhat more recently, it was shown [GHP, Theorem 2] that F itself must have a radial limit at each point of a dense subset of the boundary, if $F \in \mathfrak{B}_o$. As noted in [GHP], this would follow from the result of Rajchman and Zygmund if the latter were true for a general (complex-valued) element of λ^*, but this question has been open. In this note we show that there exists an $f \in \lambda^*$ that is nowhere differentiable and that, in fact, satisfies a Hölder condition of order one at no point.

It turns out that the existence of a nowhere differentiable $f \in \lambda^*$ is also one of various results in [MAK2], including the fact that if $f \in \lambda^*$ and is either real-valued or extends to a function holomorphic in the disc then the set of points where f is differentiable must have Hausdorff dimension 1. (The results in [MAK2] are proved in more detail in [MAK1], in particular, cf. [MAK1, Theorem 5.5].) It seems that the extremely simple argument below may nonetheless be of some independent interest: If u is an appropriate (real-valued) lacunary trigonometric series then $u \in \lambda^*$ and u is differentiable only on a set of measure zero. Now one may construct $v \in \lambda^*$ so that $f = u + iv$ is nowhere differentiable (in particular, we do not require the main technical device in [MAK2]—a characterization of the dyadic martingales arising from elements of λ^*).

1. THEOREM

The notation $\lambda^*(T)$ refers to the "little-o" *Zygmund class* on the unit circle T: we write $f \in \lambda^*(T)$ if f is a continuous (complex-valued) function on T

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and

$$\lim_{h\to 0} h^{-1}|f(e^{i(t-h)}) - 2f(e^{it}) + f(e^{i(t+h)})| = 0,$$

uniformly in t (the functions in λ^* are called "smooth functions" in [ZY]). We set

$$Mf(t) = \sup_{h>0} h^{-1}|f(e^{i(t+h)}) - f(e^{it})|,$$

so that f satisfies a Hölder condition of order 1 at e^{it} if and only if $Mf(t) < \infty$.

Theorem. *There exists* $f \in \lambda^*(T)$ *such that* $Mf(t) \equiv \infty$.

We will set $f = u + iv$, where $u \in \lambda^*$ is a (real-valued) lacunary series with $Mu = \infty$ a.e. It is impossible to achieve $Mu \equiv \infty$ here, but the following proposition will provide us with a real-valued function $v \in \lambda^*$ such that $Mv = \infty$ at every point of the set where $Mu < \infty$.

Proposition. *Suppose* $E \subset T$ *is an* F_σ *of (Lebesgue) measure zero. Then there exists a real-valued* $v \in \lambda^*(T)$ *such that* $(d/dt)v(e^{it}) = \infty$ *for every* $t \in E$.

This will follow from the following lemma. The notation $\mathrm{VMO}(T)$ refers to the space of functions of vanishing mean oscillation, as usual.

Lemma. *Suppose* E *is as in the proposition. There exists* $\varphi \in \mathrm{VMO}(T)$ *such that* $\varphi \geq 0$ *on* T *and* $\lim_{s\to t} \varphi(e^{is}) = \infty$ *for every* $e^{it} \in E$.

Proof. If we can prove the lemma for compact E then the general case follows because $\varphi \geq 0$. Suppose $E \subset T$ is a compact set of measure zero.

This implies that E is a peak set for the disc algebra: there exists a function g that is holomorphic in the unit disc D and continuous on $\overline{D}$, such that $g(e^{it}) = 1$ for $e^{it} \in E$, while $|g(z)| < 1$ for $z \in \overline{D}\backslash E$.

Now let $\Omega = \{x + iy : x > 1, |y| < 1/x\}$ and let $\psi: D \to \Omega$ be holomorphic and surjective. A theorem of Carathéodory shows that ψ extends to a homeomorphism $\overline{\psi}: \overline{D} \to \overline{\Omega}$, where $\overline{\Omega}$ denotes the closure of Ω on S, the Riemann sphere; we may take $\overline{\psi}(1) = \infty$.

Thus $G = \overline{\psi} \circ g: \overline{D} \to S$ is continuous. Let $\varphi = \mathrm{Re}(G)$. Then φ (restricted to T) is a continuous map from T to $[0, \infty]$ such that $\varphi(e^{it}) = \infty$ for $e^{it} \in E$. We only need to show that $\varphi \in \mathrm{VMO}$, but $\varphi \in \mathrm{VMO}$ because φ is the harmonic conjugate of a continuous function: The point to our choice of Ω was that $\mathrm{Im}(z) \to 0$ as z tends to ∞ within Ω, and this shows that $\mathrm{Im}(G) \in C(T)$. $\square$

Proof of the proposition. Given an F_σ set $E \subset T$ of measure zero, choose φ as in the lemma. Now define $\varphi_1 = \varphi - c$, where $c = (2\pi)^{-1}\int_0^{2\pi} \varphi(e^{it})\, dt$, and let v be an absolutely continuous function such that $(d/dt)v(e^{it}) = \varphi_1(e^{it})$ almost everywhere. It follows that $(d/dt)v(e^{it}) = \infty$ for $t \in E$, while the fact that $\varphi \in \mathrm{VMO}$ implies that $v \in \lambda^*$. $\square$

Proof of the theorem. Choose a sequence $a_j \geq 0$ with $\lim_{j\to\infty} a_j = 0$ but $\sum_{j=1}^{\infty} a_j^2 = \infty$, and set

$$u(e^{it}) = \sum_{j=1}^{\infty} 2^{-j} a_j \cos(2^j t).$$

Now the fact that $a_j \to 0$ shows that $u \in \lambda^*$ [ZY, Theorem 4.10, p. 47], while $\sum_{j=1}^{\infty} a_j^2 = \infty$ shows that $Mu(e^{it}) = \infty$ for almost all t. This will be "clear" to readers with some experience dealing with lacunary series; a proof is already at least implicit in [ZY]:

Let $d_N(t) = -\sum_{j=1}^{N} a_j \sin(2^j t)$. Then it is well known that $(d_N(t))$ is unbounded for almost every value of t [ZY, Theorem 6.4, p. 203 and Remark (c), p. 205]. But it is easy to obtain a uniform upper bound on the quantity

$$h_N^{-1}[u(e^{i(t+h_N)}) - u(e^{it})] - d_N(t)$$

if $h_N = 2^{-N}\pi$, so that $Mu = \infty$ at any point where (d_N) is unbounded.

Now let $E = \{e^{it} : Mu(e^{it}) < \infty\}$. We have just seen that E has measure zero. Continuity of $u \in \lambda^*$ shows that $\{Mu \leqslant j\}$ is closed for $j = 1, 2, \ldots$, so that E is an F_σ. Choose v as in the proposition and let $f = u + iv$. Then $f \in \lambda^*$ and $Mf \equiv \infty$. □

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