LINEAR DIFFERENTIAL EQUATIONS WITH EXCEPTIONAL FUNDAMENTAL SETS. II

NORBERT STEINMETZ

(Communicated by Clifford J. Earle, Jr.)

ABSTRACT. We prove a sharp order estimate for entire functions of completely regular growth, whose zeros are distributed near finitely many rays $\arg z = \omega_j$ in terms of the angles ω_j . This result then leads immediately to a proof of a conjecture of Hellerstein and Rossi concerning the distribution of zeros of the solutions of linear differential equations with polynomials coefficients.

1. INTRODUCTION

Let w_1, w_2 be linearly independent solutions of

$$(1) \quad w'' + p(z)w = 0,$$

where p is a nonconstant polynomial. From the well-known results of Hille [6, Chapter 5] on the asymptotic distribution of zeros, Gundersen [4] deduced that the exponent of convergence of the nonreal zeros of $E = w_1 w_2$ equals $1 + \frac{1}{2} \deg p$. In an earlier paper, Hellerstein, Shen, and Williamson [5] showed that E has infinitely many nonreal zeros.

The methods in [4, 5] do not apply to the n th order case. In this paper we prove an analogon of Gundersen's result for the solutions of n th order linear differential equations

$$(2) \quad w^{(n)} + p_{n-2}(z)w^{(n-2)} + \cdots + p_0(z)w = 0$$

with polynomial coefficients, which is even slightly sharper. In particular, we solve Problem 2.72 in [1], posed by Hellerstein and Rossi.

This is done by proving a sharp order estimate (Theorem 1) for entire functions of completely regular growth, which seems to be of independent interest.

2. NOTATION

Let E be an entire function of finite positive order λ and denote the counting function of zeros of E lying in the planar set S by $N(r, 1/E, S)$. We say that the zeros of E are distributed near the rays

$$(3) \quad \arg z = \omega_j, \quad 0 \leq \omega_1 < \omega_2 < \cdots < \omega_k < \omega_{k+1} = \omega_1 + 2\pi,$$

Received by the editors June 9, 1990 and, in revised form, December 12, 1990.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 30D35, 34A20.

if for every sufficiently small $\delta > 0$,

$$(4) \quad N(r, 1/E, \omega_j + \delta < \arg z < \omega_{j+1} - \delta) = o(r^\lambda) \quad \text{as } r \rightarrow \infty$$

($j = 1, 2, \dots, k$).

The basic results in [8] for solutions of equation (2) can be summarized as follows:

Let w be a transcendental solution of (2), where $p_0, \dots, p_{n-2}$ are polynomials. Then w is a completely regular growing entire function of order λ (in the sense of Lewin [7]), $0 < \lambda < \infty$, whose zeros are distributed near finitely many rays (3).

3. RESULTS

We first prove a result on completely regular growing entire functions with radially distributed zeros, from which the main result concerning the distribution of zeros of a fundamental set of (3) follows.

Theorem 1. *Let E be a completely regular growing entire function of order λ , whose zeros are distributed near the rays (3). Then either*

$$(5) \quad \lambda \leq \frac{\pi}{\min(\omega_{j+1} - \omega_j)}$$

or else

$$(6) \quad \delta(0, E) > 0.$$

Remark. There is a similar result [3, Corollary 1.2]: If the zeros of the entire function E are lying on the rays $\arg z = \omega_j$ and if the order of E is finite, but sufficiently large (depending on the geometry of the rays), then $\delta(0, E) > 0$. For us it is important to have an explicit (and sharp) bound.

Theorem 2. *Let $\{w_1, w_2, \dots, w_n\}$ be a fundamental set of (2), where $p_0, \dots, p_{n-2}$ are polynomials. Suppose that the zeros of $\{w_1, \dots, w_n\}$ are distributed near the rays (3). Then either*

$$(7) \quad \lambda(w_\nu) \leq \frac{\pi}{\min(\omega_{j+1} - \omega_j)}, \quad \nu = 1, \dots, n,$$

or else the coefficients in (2) are constants.

Remark. The product $E = w_1 w_2 \cdots w_n$ is a completely regular growing entire function of order λ , $0 < \lambda < \infty$, or a polynomial, and the radial distribution of the zeros of $\{w_1, \dots, w_n\}$ corresponds to the function E (and its order).

For the sake of completeness we reformulate Theorem 2 in two special cases.

Corollary 1. *Let $\{w_1, \dots, w_n\}$ be a fundamental set of (2), whose zeros are distributed near the rays $\arg z = 2j\pi/k$ ($1 \leq j \leq k$, $k \geq 2$). Then*

$$(8) \quad \lambda(w_\nu) \leq k/2, \quad \nu = 1, \dots, n.$$

Corollary 2. *Let $\{w_1, \dots, w_n\}$ be a fundamental set of (2), whose zeros are distributed near the real axis. Then the coefficients in (2) are constants.*

Remark. This result has also been proven by Brüggemann [2] in a slightly weaker form.

4. PROOF OF THEOREM 1

Since the function E has completely regular growth and since its zeros are distributed near the rays $\arg z = \omega_j$, there are complex numbers c_j such that

$$(9) \quad \log |E(z)| = \operatorname{Re}(c_j z^\lambda) + o(|z|^\lambda).$$

This is true as $|z| \rightarrow \infty$ outside a set $\mathcal{E} \subseteq (0, \infty)$ of linear density zero:

$$(10) \quad \lim_{r \rightarrow \infty} \operatorname{mes}(\mathcal{E} \cap [0, r])/r = 0$$

and uniformly in $\omega_j + \eta \leq \arg z \leq \omega_{j+1} - \eta$, $\eta > 0$ arbitrary.

Thus the contribution of the sector $\omega_j < \arg z < \omega_{j+1}$ to $m(r, E)$ and $m(r, 1/E)$ is asymptotically $M_j r^\lambda$ and $m_j r^\lambda$, respectively, where

$$(11) \quad M_j = \frac{1}{2\pi} \int_{\omega_j}^{\omega_{j+1}} [\operatorname{Re}(c_j e^{i\lambda\theta})]^+ d\theta$$

and

$$(12) \quad m_j = \frac{1}{2\pi} \int_{\omega_j}^{\omega_{j+1}} [\operatorname{Re}(c_j e^{i\lambda\theta})]^- d\theta$$

(as usual $x^+ = \max(0, x)$ and $x^- = \max(0, -x)$).

If we assume $\delta(0, E) = 0$, then we must have $m_j = 0$ ($1 \leq j \leq k$), but $M_j \neq 0$ for at least one j since E has order λ .

Thus $c_j \neq 0$ and $|c_j| \cos(\arg c_j + \lambda\theta) \geq 0$ in $\omega_j < \theta < \omega_{j+1}$ for at least one j , which gives

$$(13) \quad \lambda(\omega_{j+1} - \omega_j) \leq \pi$$

and so (7).

5. PROOF OF THEOREM 2

Set $E = w_1 w_2 \cdots w_n$ and assume first that E is transcendental of order λ . The Wronskian

$$(14) \quad W = W(w_1, w_2, \dots, w_n)$$

is a constant c , say. Thus

$$(15) \quad m\left(r, \frac{1}{E}\right) = m\left(r, \frac{W}{E}\right) + O(1) = O(\log r)$$

by the lemma on the logarithmic derivative and so

$$(16) \quad \lambda \leq \pi / \min(\omega_{j+1} - \omega_j)$$

holds by Theorem 1.

However, if $\lambda < \max(\lambda(w_1), \dots, \lambda(w_n))$, then by [9, Theorem 2] the coefficients of (2) are constants, and this is of course also true if E is a polynomial. Thus, either $\lambda = \max(\lambda(w_1), \dots, \lambda(w_n))$ and (16) implies (7) or the coefficients of (2) are constants.

REFERENCES

1. D. A. Brannan and W. K. Hayman, *Research problems in complex analysis*, Bull. London Math. Soc. **21** (1989), 1–35.
2. F. Brüggemann, *On solutions of linear differential equations with real zeros; proof of a conjecture of Hellerstein and Rossi*, Proc. Amer. Math. Soc. **113** (1991), 371–379.
3. A. Edrei, W. H. J. Fuchs, and S. Hellerstein, *Radial distribution of the values of a meromorphic function*, Pacific J. Math. **11** (1961), 135–151.
4. G. Gundersen, *On the real zeros of solutions of $f'' + A(z)f = 0$ where $A(z)$ is entire*, Ann. Acad. Sci. Fenn. Ser. A. I Math. **11** (1986), 275–294.
5. S. Hellerstein, L. C. Shen, and J. Williamson, *Real zeros of derivatives of meromorphic functions and solutions of second-order differential equations*, Trans. Amer. Math. Soc. **285** (1984), 759–776.
6. E. Hille, *Lectures on ordinary differential equation*, Wiley, New York, 1976.
7. B. J. Lewin, *Nullstellenverteilung ganzer Funktionen*, Akademie-Verlag, Berlin, 1962.
8. N. Steinmetz, *Exceptional values of solutions of linear differential equations*, Math. Z. **201** (1989), 317–326.
9. ———, *Linear differential equations with exceptional fundamental sets*, submitted.

INSTITUT FÜR MATHEMATIK, UNIVERSITÄT DORTMUND, POSTFACH 500 500, D-4600 DORTMUND,
GERMANY