A WHITNEY STRATIFICATION AND EQUISINGULAR FAMILY OF QUASI-ORDINARY SINGULARITIES

CHUNSHENG BAN

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ABSTRACT. Let $(V, 0) \subset (\mathbb{C}^{d+1}, 0)$ be a quasi-ordinary singularity and $\pi: (V, 0) \rightarrow (\mathbb{C}^d, 0)$ a quasi-ordinary projection. $\mathbb{C}^d$ has a natural Whitney stratification given by the multiplicities of the discriminant locus of π . It is proved that the pullback of this stratification gives a Whitney stratification of $(V, 0)$. Then using this result, an equisingular family of quasi-ordinary singularities is studied.

1. WHITNEY STRATIFICATION

Let $(V, 0)$ be an irreducible hypersurface germ in $(\mathbb{C}^{d+1}, 0)$ and assume that $0 \in V$ is a singular point (which is not isolated in general). $(V, 0)$ is quasi-ordinary if there is a finite map of analytic germs $\pi: (V, 0) \rightarrow (\mathbb{C}^d, 0)$ that induces a surjection of Zariski tangent spaces and whose discriminant locus D (the hypersurface in $\mathbb{C}^d$ over which π ramifies) has a normal crossing. π is called a quasi-ordinary projection for $(V, 0)$. We may choose local coordinates $x_1, x_2, \dots, x_d, z$ such that $\pi(x_1, \dots, x_d, z) = (x_1, \dots, x_d)$ and $(V, 0) \subset (\mathbb{C}^{d+1}, 0)$ is defined by a pseudopolynomial f in Z :

$$f(Z) = Z^m + g_1(X_1, \dots, X_d)Z^{m-1} + \dots + g_m(X_1, \dots, X_d)$$

where the g_i are power series in $X_1, \dots, X_d$ and $g_i(0, \dots, 0) = 0$. $(V, 0)$ being quasi-ordinary means that the discriminant Δ of f has the form

$$\Delta = X_1^{k_1} \dots X_d^{k_d} u(X_1, \dots, X_d), \quad u(0, \dots, 0) \neq 0.$$

It is known that the roots of $f(Z) = 0$ can be represented by fractional power series (cf. [A])

$$\zeta_i = H_i(X_1^{1/n}, \dots, X_d^{1/n}) = H(\omega_{i1}X_1^{1/n}, \dots, \omega_{id}X_d^{1/n}), \quad 1 \leq i \leq m,$$

where H is a power series and the ω_{ij} are n th roots of unity. Then we have a parametrization of $(V, 0)$ by $\zeta = H(X_1^{1/n}, \dots, X_d^{1/n})$, i.e., $(V, 0)$ is the

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image of the map $\Phi: U \rightarrow \mathbb{C}^{d+1}$ (U some neighborhood of 0 in $\mathbb{C}^d$) given by

$$(*) \quad \Phi(x_1, \dots, x_d) = (x_1^n, \dots, x_d^n, H(x_1, \dots, x_d)).$$

ζ is called a quasi-ordinary parametrization of $(V, 0)$.

Since the discriminant is $\Delta = \prod_{i \neq j} (\zeta_i - \zeta_j)$, unique factorization of (fractional) power series gives

$$\zeta_i - \zeta_j = X_1^{a_1/n} \cdots X_d^{a_d/n} u_{ij}(X_1^{1/n}, \dots, X_d^{1/n})$$

where $u_{ij}(0, \dots, 0) \neq 0$ and a_l depends on i, j ($1 \leq l \leq d$).

The fractional monomials $M_{ij} = X_1^{a_1/n} \cdots X_d^{a_d/n}$ so obtained are called the characteristic monomials of $\zeta = H(X_1^{1/n}, \dots, X_d^{1/n})$. This set of monomials determines quite a lot of geometry and topology of $(V, 0)$ (cf. [B, G, L1, L2]).

Let $\pi: (V, 0) \rightarrow (\mathbb{C}^d, 0)$ and D be as before and $|D|$ be the reduced discriminant locus of π . Then $\mathbb{C}^d$ has a filtration

$$D_0 = \mathbb{C}^d \supset D_1 = |D| \supset D_2 \supset \cdots$$

where $D_i = \{x \in \mathbb{C}^d \mid \mu_{|D|}(x) \geq i\}$ and $\mu_{|D|}(x)$ is the multiplicity of $|D|$ at x .

Proposition 1. *The stratification of $\mathbb{C}^d$ associated to the above filtration is a Whitney stratification.*

Proof. Left to the reader. Q.E.D.

Relabeling the variables if necessary, we may assume that the discriminant Δ of $f(Z)$ is divisible by X_i if and only if $i \leq c$. Then $|D|$ is defined by $X_1 \cdots X_c = 0$ and $(x_1, \dots, x_d) \in D_i$ if and only if at least i of $x_1, \dots, x_c$ are 0. Let

$$A_I = \{(x_1, \dots, x_d) \in \mathbb{C}^d \mid x_i = 0 \text{ for } i \in I\}, \quad I \subset \{1, 2, \dots, c\},$$

and $A_I^\circ = A_I - \bigcup_{I \subsetneq J} A_J$. Then the connected components of $D_i - D_{i+1}$ are the A_I° with $I \subset \{1, 2, \dots, c\}$ and $|I| = i$. Let $V_I^\circ = \pi^{-1}(A_I^\circ)$ and $V_I = \pi^{-1}(A_I)$ be the pullbacks of A_I° and A_I , respectively. Since $\pi|_{V_I^\circ}: V_I^\circ \rightarrow A_I^\circ$ is an étale covering (cf. [L2, Proposition 7.1.2, p. 86]), $V = \bigcup V_I^\circ$ is a complex analytic stratification. The following theorem says it is also a Whitney stratification.

Theorem 2. $V = \bigcup V_I^\circ$ is a Whitney stratification.

Proof. Let V_I°, V_J° be two strata such that $V_I^\circ \subset \overline{V_J^\circ} = V_J$ and $p \in V_I^\circ$. To show that (V_J°, V_I°) satisfies Whitney conditions at p , we begin with some reduction.

By the characterization of the characteristic monomials given by Lipman (cf. [L2, Proposition 1.5]), V_J is a quasi-ordinary singularity with quasi-ordinary parametrization ζ_J , where ζ_J is the fractional power series obtained from ζ by substituting 0 for every one of the X_i , $i \in J$. So we may assume that $J = \emptyset$, i.e., $V_J = V$. Without loss of generality, we may assume that $I = \{1, 2, \dots, e\} \subset \{1, 2, \dots, c\}$, $e \leq c$. So we have to check the Whitney conditions for $(V_J^\circ, V_I^\circ) = (V^\circ, V_{\{1, 2, \dots, e\}}^\circ)$ at $p \in V_{\{1, 2, \dots, e\}}^\circ$ where $V^\circ = V_\emptyset^\circ$.

By a lemma of Lipman and its proof (cf. [L2, Lemma 7.2.3]), there exists an analytic automorphism $\psi: (\mathbb{C}^{d+1}, p) \rightarrow (\mathbb{C}^{d+1}, 0)$ that maps (V, p) onto

a subgerm $(V', 0)$ of $(\mathbb{C}^{d+1}, 0)$ such that $(V', 0)$ has quasi-ordinary parametrization

$$\zeta' = H'(X_1^{1/n}, \dots, X_e^{1/n}, X_{e+1}, \dots, X_d)$$

that has no characteristic monomial (in this case, $(V', 0)$ is smooth) or whose smallest characteristic monomial has the form

$$X_1^{a_1/n} \dots X_e^{a_e/n}, \quad a_i \text{ integers.}$$

Under this automorphism, $(V_{\{1,2,\dots,e\}}, p)$ is mapped onto a subgerm $(Y, 0)$ of $(V', 0)$ where Y is defined by $X_1 = \dots = X_e = 0$ in $(V', 0)$. Then ζ' can be written as

$$\zeta' = X_1^{a_1/n} \dots X_e^{a_e/n} \varepsilon(X_1^{1/n}, \dots, X_d) + H_0(X_1, \dots, X_d)$$

where $\varepsilon(0, \dots, 0) \neq 0$ and $H_0(X_1, \dots, X_d)$ is the sum of all those terms $cX_1^{\rho_1} \dots X_d^{\rho_d}$ ($c \neq 0$) appearing in ζ' for which every exponent ρ_i is an integer. By the analytic automorphism

$$(x_1, \dots, x_d, z) \rightarrow (x_1, \dots, x_d, z - H_0(x_1, \dots, x_d))$$

of the germ $(\mathbb{C}^{d+1}, 0)$, $(V', 0)$ is mapped onto a quasi-ordinary hypersurface germ with quasi-ordinary parametrization

$$\zeta' - H_0 = X_1^{a_1/n} \dots X_e^{a_e/n} \varepsilon(X_1^{1/n}, \dots, X_d), \quad \varepsilon(0, \dots, 0) \neq 0.$$

Therefore, it suffices to check the Whitney conditions for (V°, Y) at $0 \in Y$ where V has a quasi-ordinary parametrization

$$\zeta = X_1^{a_1/n} \dots X_e^{a_e/n} \varepsilon(X_1^{1/n}, \dots, X_d), \quad \varepsilon(0, \dots, 0) \neq 0.$$

Then $(*)$ can be written as

$$(**) \quad \Phi(x_1, \dots, x_d) = (x_1^n, \dots, x_e^n, x_{e+1}, \dots, x_d, x_1^{a_1} \dots x_e^{a_e} \varepsilon(x_1, \dots, x_d))$$

and $V^\circ = V_\emptyset^\circ = \{(x_1, \dots, x_d, z) \in V \mid x_1, \dots, x_e \neq 0\}$. Let

$$P_k = \Phi(x_{k1}, \dots, x_{kd}) \in V^\circ, \quad Q_k = (0, \dots, 0, y_{ke+1}, \dots, y_{kd}, 0) \in Y,$$

such that $P_k \rightarrow 0$, $Q_k \rightarrow 0$. Suppose $T_{P_k} = T_{P_k}V \rightarrow T$ and $P_kQ_k \rightarrow L = (L_1: \dots: L_{d+1})$. We need to show that $L \subset T$.

By abuse of notation, we also use T_{P_k} , T to indicate the direction of T_{P_k} , T respectively.

The parametrization $(**)$ gives $T_{P_k} = ((-1)^1 M_1, \dots, (-1)^{d+1} M_{d+1})$ where M_i is the minor of the Jacobian matrix $D\Phi(P_k)$ with the i th column deleted. So we have

$$T_{P_k} = \left(\frac{x_k^a}{n x_{k1}^n} \left(a_1 \varepsilon + x_{k1} \frac{\partial \varepsilon}{\partial x_1} \right), \dots, \frac{x_k^a}{n x_{ke}^n} \left(a_e \varepsilon + x_{ke} \frac{\partial \varepsilon}{\partial x_e} \right), \right. \\ \left. x_k^a \frac{\partial \varepsilon}{\partial x_{e+1}}, \dots, x_k^a \frac{\partial \varepsilon}{\partial x_d}, -1 \right)$$

where $x_k^a = x_{k1}^{a_1} \dots x_{ke}^{a_e}$. The secant P_kQ_k is given by

$$P_kQ_k = (x_{k1}^n, \dots, x_{ke}^n, x_{ke+1} - y_{ke+1}, \dots, x_{kd} - y_{kd}, x_k a_\varepsilon).$$

Write $T = (b_1, \dots, b_{d+1})$. Then $T_{P_k} \rightarrow T$ and $P_kQ_k \rightarrow L$ mean that there exist $\lambda_k, \mu_k \in \mathbb{C}$ such that

$$\lim \lambda_k T_{P_k} = T, \quad \lim \mu_k P_kQ_k = L.$$

Writing the two limits componentwise, we get

$$(1) \quad \lim \lambda_k \frac{x_k^a}{n x_{kj}^n} \left(a_j \varepsilon + x_{kj} \frac{\partial \varepsilon}{\partial x_j} \right) = b_j, \quad 1 \leq j \leq e,$$

$$(2) \quad \lim \lambda_k x_k^a \frac{\partial \varepsilon}{\partial x_j} = b_j, \quad e < j \leq d,$$

$$(3) \quad \lim(-\lambda_k) = b_{d+1},$$

$$(1') \quad \lim \mu_k x_{kj}^n = L_j, \quad 1 \leq j \leq e,$$

$$(2') \quad \lim \mu_k (x_{kj} - y_{kj}) = L_j, \quad e < j \leq d,$$

$$(3') \quad \lim \mu_k x_k^a \varepsilon = L_{d+1}.$$

Then

$$\begin{aligned} T \cdot L &= b_1 L_1 + \cdots + b_{d+1} L_{d+1} = \lim \lambda_k T_{P_k} \cdot \mu_k P_k Q_k \\ &= \lim \left(\sum_{j=1}^e \frac{\lambda_k \mu_k x_k^a}{n} \left(a_j \varepsilon + x_{kj} \frac{\partial \varepsilon}{\partial x_j} \right) \right. \\ (4) \quad &\quad \left. + \sum_{j=e+1}^d \lambda_k \mu_k x_k^a (x_{kj} - y_{kj}) \frac{\partial \varepsilon}{\partial x_j} - \lambda_k \mu_k x_k^a \varepsilon \right) \\ &= -b_{d+1} L_{d+1} \left(\sum_{j=1}^e \frac{a_j}{n} - 1 \right). \end{aligned}$$

If $b_{d+1} L_{d+1} = 0$, then $T \cdot L = 0$. If $b_{d+1} L_{d+1} \neq 0$, then we may assume that $b_{d+1} = -1$, $L_{d+1} = \varepsilon(0)$ and take $\lambda_k = 1$, $\mu_k = 1/x_k^a$. Then (1) and (1') show that both limits $\lim x_k^a/x_{kj}^n$ and $\lim x_{kj}^n/x_k^a$ exist for $1 \leq j \leq e$, and so the limits $\lim (x_k^a/x_{kj}^n)^{a_j}$ and $\lim (x_{kj}^n/x_k^a)^{a_j}$ exist for $1 \leq j \leq e$. Then the limits

$$\begin{aligned} \lim \prod_{j=1}^e \left(\frac{x_k^a}{x_{kj}^n} \right)^{a_j} &= \lim (x_k^a)^{(a_1 + \cdots + a_e) - n}, \\ \lim \prod_{j=1}^e \left(\frac{x_{kj}^n}{x_k^a} \right)^{a_j} &= \lim (x_k^a)^{n - (a_1 + \cdots + a_e)} \end{aligned}$$

exist. Since $\lim x_k^a = 0$, we must have $n - (a_1 + \cdots + a_e) = 0$, i.e., $\sum_{j=1}^e a_j/n - 1 = 0$. We still have $T \cdot L = 0$ in (4). Therefore $L \subset T$. This completes the proof. Q.E.D.

2. EQUISINGULAR FAMILY

As an application, we now study the equisingularity of a family of quasi-ordinary singularities.

Let $\{(V_t, 0_t) | t \in S\}$ be a family of d -dimensional hypersurface germs parametrized by a smooth germ (S, s) of dimension r . Let V be the total space and W be the locus of the origins 0_t . Namely, there is a flat morphism $\phi: (V, 0) \rightarrow (S, s)$, where $0 = 0_s$, of complex analytic germs and a section $i: (S, s) \rightarrow (V, 0)$ of ϕ such that $V_t = \phi^{-1}(t)$ and $i(t) = 0_t$. Then $W = i(S)$

is a smooth subspace of V that is isomorphic to S . We have the following notions of equisingularity (cf. [Z]):

- V is topologically equisingular along W at 0 if all $(V_t, 0_t)$ have the same topological type for t close to s , where we say that two germs $(V_1, 0_1)$ and $(V_2, 0_2)$ in $(\mathbb{C}^{d+1}, 0)$ have the same topological type if there exists a local homeomorphism $\rho: (\mathbb{C}^{d+1}, V_1, 0_1) \rightarrow (\mathbb{C}^{d+1}, V_2, 0_2)$ of the triples.
- V is differentially equisingular along W at 0 if (V, W) satisfies the Whitney conditions at 0.

Now, we assume that there is a finite map $p: V \rightarrow \mathbb{C}^d \times S$ such that $p(W) = \{0\} \times S$ is an isomorphism and $p_t = p|_{V_t}: V_t \rightarrow \mathbb{C}^d$ is a quasi-ordinary projection of $(V_t, 0_t)$ for every $t \in S$. Thus $\{(V_t, 0_t)\}$ is a family of quasi-ordinary singularities. Then by the results of Gau and Lipman (cf. [G, L2]), we have

Theorem 3. *V is topologically equisingular along W at each point of W if all $(V_t, 0_t)$ have the same characteristic monomials. Conversely, $(V_t, 0_t)$ have the same (normalized) characteristic monomials (may not be with respect to the p_t) if V is topologically equisingular along W .*

Therefore, the characteristic monomials determine the topological equisingularity of the family $\{(V_t, 0_t)\}$. Similar results are also true for differential equisingularity. We will prove the following theorems.

Theorem 4. *If all $(V_t, 0_t)$ have the same characteristic monomials then W is a stratum of a Whitney stratification of $(V, 0)$, and so $(V, 0)$ is differentially equisingular along W at any point of W .*

Theorem 5. *If W is a stratum of a Whitney stratification of $(V, 0)$ then all the $(V_t, 0_t)$ have the same (normalized) characteristic monomials.*

Proof of Theorem 5. By a well-known result of Thom and Mather, $(V, 0)$ is topologically equisingular along W at each point of W . Then all $(V_t, 0_t)$ have the same topological type in $(\mathbb{C}^{d+1}, 0)$. By Theorem 2, all $(V_t, 0_t)$ have the same (normalized) characteristic monomials. Q.E.D.

To prove Theorem 4, we start with some preparation.

Lemma 6. *Let $F: (X, x) \rightarrow (S, s)$ be a deformation of (X_0, x_0) with a section $\sigma: S \rightarrow X$, and let $X_t = F^{-1}(t)$ and $x_t = \sigma^{-1}(t)$. If (X_t, x_t) is analytically isomorphic to (X_0, x_0) for all $t \in (S, s)$, then F is analytically trivial, i.e., X is isomorphic to $(X_0, x_0) \times (S, s)$ (cf. [T, Theorem 2.1]).*

Since (S, s) is smooth, we may assume that $(S, s) = (\mathbb{C}^r, 0)$. By the flatness of $\phi: (V, 0) \rightarrow (\mathbb{C}^r, 0)$, we may assume that $V \subset \mathbb{C}^{d+1} \times \mathbb{C}^r$ is defined by

$$f(X_1, \dots, X_{d+1}, Y_1, \dots, Y_r) = f(X, Y) = 0$$

and ϕ is induced by the second projection $\mathbb{C}^{d+1} \times \mathbb{C}^r \rightarrow \mathbb{C}^r$. The $(V_t, 0_t)$ is defined by $f_t(X) = f(X_1, \dots, X_{d+1}, t_1, \dots, t_r) = 0$ in $\mathbb{C}^{d+1}$. Since $(V_0, 0)$ is a quasi-ordinary singularity, we may choose the local coordinates $x_1, \dots, x_d, z$ for $\mathbb{C}^{d+1}$ at 0 such that

$$\begin{aligned} f_0(X_1, \dots, X_d, Z) &= f(X_1, \dots, X_d, Z, 0, \dots, 0) \\ &= Z^m + g_1(X_1, \dots, X_d)Z^{m-1} + \dots + g_m(X_1, \dots, X_d) \end{aligned}$$

as before. Then $f(0, \dots, 0, Z, 0, \dots, 0) = Z^m$. By the Weierstrass preparation theorem, we may assume

$$f(X_1, \dots, X_d, Z, Y_1, \dots, Y_r) = Z^m + a_1(X, Y)Z^{m-1} + \dots + a_m(X, Y).$$

Also, we may identify $p: V \rightarrow \mathbb{C}^d \times S = \mathbb{C}^d \times \mathbb{C}^r$ with the restriction to V of

$$\pi \times \text{id}: \mathbb{C}^{d+1} \times \mathbb{C}^r \rightarrow \mathbb{C}^d \times \mathbb{C}^r$$

where $\pi(x_1, \dots, x_d, z) = (x_1, \dots, x_d)$. Now, we can prove the following key lemma.

Lemma 7. *If all $(V_t, 0_t)$ have the same characteristic monomials with respect to p_t , then the total space $(V, 0)$ itself is a quasi-ordinary singularity with the quasi-ordinary projection $p: V \rightarrow \mathbb{C}^d \times S$.*

Proof. Let $\Delta = \Delta(X, Y)$ be the discriminant of f (as a polynomial in Z). Then $\Delta_t = \Delta(X, t)$ is the discriminant of $f_t(X_1, \dots, X_d, Z)$. Since all $(V_t, 0_t)$ have the same characteristic monomials, $\Delta_t = \Delta_0 u_t$, where u_t is unit, for $t \in (\mathbb{C}^r, 0)$; and so the discriminant locus (D_t, t) of p_t is analytically isomorphic to $(D_0, 0)$. If D is the discriminant locus of p , then Lemma 6 asserts that p is analytically trivial, and so $D \simeq D \times \mathbb{C}^r$ is a normal crossing. Therefore, $(V, 0)$ itself is a quasi-ordinary singularity. Q.E.D.

Now, we can prove Theorem 4.

Proof of Theorem 4. By Lemma 7 and its proof, $(V, 0)$ is a quasi-ordinary singularity defined by $f(X_1, \dots, X_d, Z, Y_1, \dots, Y_r) = 0$, where

$$f(0, \dots, 0, Y_1, \dots, Y_r) \equiv 0,$$

with discriminant

$$(5) \quad \Delta = X_1^{a_1} \dots X_e^{a_e} u(X_1, \dots, X_d, \dots, Y_1, \dots, Y_r), \quad u(0, \dots, 0) \neq 0.$$

Let $V = \bigcup V_I^\circ$ be the Whitney stratification as in Theorem 2.

If $e = d$ in (5), $W = 0 \times \mathbb{C}^r = V_{\{1, 2, \dots, d\}} = V_{\{1, 2, \dots, d\}}^\circ$; and so W is a stratum of $V = \bigcup V_I^\circ$.

If $e < d$ in (5), then by (**), $V_{\{1, 2, \dots, e\}}^\circ = \{0\} \times \mathbb{C}^{d+1-e} \times \mathbb{C}^r$ and we have

$$W = 0 \times \mathbb{C}^r \subset V_{\{1, 2, \dots, e\}}^\circ = (0 \times \mathbb{C}^{d+1-e}) \times \mathbb{C}^r.$$

It is trivial that $(V_{\{1, 2, \dots, e\}}^\circ - W, W)$ satisfies Whitney conditions. For any stratum V_I° with $|I| < e$, since $(V_I^\circ, V_{\{1, 2, \dots, e\}}^\circ)$ satisfies Whitney conditions and $W \subset V_{\{1, 2, \dots, e\}}^\circ$, (V_I°, W) and $(V_I^\circ, V_{\{1, 2, \dots, e\}}^\circ - W)$ satisfy Whitney conditions. Then we have a Whitney stratification

$$V = \left(\bigcup_{|I| < e} V_I^\circ \right) \cup (V_{\{1, 2, \dots, e\}}^\circ - W) \cup W$$

with W as a stratum. This completes the proof. Q.E.D.

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REFERENCES

- [A] S. S. Abhyankar, *On the ramification of algebraic functions*, Amer. J. Math. **77** (1955), 575–592.
- [B] C. Ban, *Whitney stratification, equisingular family and the auréole of quasi-ordinary singularity*, Thesis, Purdue Univ., 1990.
- [G] Y. N. Gau, *Embedded topological classification of quasi-ordinary singularities*, Mem. Amer. Math. Soc., vol. 74, Amer. Math. Soc., Providence, RI, 1988, pp. 109–129.
- [HL] J.-P. G. Henry and D. T. Lé, *Limits d'espaces tangents*, Lecture Notes in Math., vol. 482, Springer-Verlag, New York, 1975, pp. 251–265.
- [L1] J. Lipman, *Quasi-ordinary singularities of embedded surfaces*, Thesis, Harvard Univ., 1965.
- [L2] ———, *Topological invariants of quasi-ordinary singularities*, Mem. Amer. Math. Soc., vol. 74, Amer. Math. Soc., Providence, RI, 1988, pp. 1–107.
- [LT] D. T. Lé and B. Teissier, *Limits d'espaces tangents en géométrie analytique*, Comment. Math. Helv. **63** (1988), 540–578.
- [T] B. Teissier, *Deformations a type topologique constant*. I, Astérisque **16** (1974), 215–249.
- [W] H. Whitney, *Tangents to an analytic variety*, Ann. of Math. (2) **81** (1964), 496–549.
- [Z] O. Zariski, *Some open questions in the theory of singularity*, Bull. Amer. Math. Soc. **77** (1971), 481–491.

DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, MANSFIELD, OHIO 44906
E-mail address: cban@mps.ohio-state.edu