A NOTE ON SOLUTIONS OF THE SCHRÖDINGER EQUATION

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(Communicated by Barbara L. Keyfitz)

ABSTRACT. Using a suitable geometric series a simple proof for the continuity of L^1 -solutions of the Schrödinger equation $Lu := \Delta u - Vu$ (V being a Kato function) is obtained. It works as well for uniformly elliptic operators having Hölder continuous coefficients and Kato measures instead of Kato functions. Moreover, it is shown that results on singularities are immediate consequences of the local existence of continuous Green functions for L .

Let Ω be an open set in $\mathbb{R}^n$ (bounded, if $n \leq 2$) and let G_Ω denote the Green function for $\frac{1}{2}\Delta$ on Ω . Fix a Kato measure μ on Ω , i.e., a signed measure μ on Ω such that the potential $G_B^{|\mu|} := \int G_B(\cdot, y)|\mu|(dy)$ is continuous and real for every ball B with $\bar{B} \subset \Omega$ (e.g., $\mu = f\lambda$, $f \in \mathcal{L}_{\text{loc}}^p(\Omega, \lambda)$, λ Lebesgue measure, $p > n/2$). If B is sufficiently small then $G_B^{s|\mu|} \leq \frac{1}{3}s$ for every positive superharmonic function s on B . In [BHH] this fact has been used to obtain a Harnack inequality for positive continuous real solutions of $Lu := \frac{1}{2}\Delta u - u\mu = 0$ on Ω (for uniform Harnack inequalities see [H]). Moreover, it led to comparison of Green functions, harmonic measures resp. for L and $\frac{1}{2}\Delta$.

The purpose of this note is to show that it is equally useful for a simple proof of continuity of $\mathcal{L}^1$ -solutions. Furthermore, it will be pointed out that results on singularities obtained in [VY] in a fairly complicated way are immediate consequences of the local existence of continuous Green functions for L .

Let us note that our results hold as well if we replace Δ by an operator of the form $A = \sum_{i,j} a_{ij}D_{ij} + \sum_i b_i D_i$ that is uniformly elliptic and has Hölder continuous coefficients a_{ij} , b_i (see the remark at the end of [BHH]).

Theorem 1. *Let $u \in \mathcal{L}^1(\Omega, \lambda + |\mu|)$ such that $u + G_\Omega^{\mu\mu}$ is $(\lambda + |\mu|)$ -a.e. equal to a harmonic function on Ω . Then there exists a function $v \in \mathcal{C}(\Omega)$ such that $u = v$ $(\lambda + |\mu|)$ -a.e.*

Before proceeding to a proof let us briefly discuss the meaning of this result. Clearly the assumption on u implies that $Lu := \frac{1}{2}\Delta u - u\mu = 0$ (in the sense of distributions) and then $Lv = 0$ as well. We note that, in general, a solution $u \in \mathcal{L}^1(\Omega, \lambda + |\mu|)$ of $Lu = 0$ may not even be in $\mathcal{L}^\infty(\Omega, \lambda)$. If, however,

Received by the editors August 21, 1990 and, in revised form, June 4, 1991.

1991 *Mathematics Subject Classification.* Primary 31D05, 35J10.

Key words and phrases. Schrödinger equation, isolated singularity, Kato function, Kato measure.

μ is absolutely continuous with respect to λ , i.e., if $\mu = V\lambda$, V a (local) Kato function, then $\lambda + |\mu|$ is equivalent to λ , $u \in \mathcal{L}^1(\Omega, \lambda + |\mu|)$ means $u, Vu \in \mathcal{L}^1(\Omega, \lambda)$, and $u + G^{u\mu}$ is λ -a.e. equal to a harmonic function on Ω if (and only if) $\frac{1}{2}\Delta u - Vu = 0$. Thus Theorem 1 implies the following result, which has been obtained in [AS] in a totally different way.

Corollary. *Let V be a (local) Kato function on Ω and $u, Vu \in \mathcal{L}^1(\Omega, \lambda)$ such that $\frac{1}{2}\Delta u - Vu = 0$. Then there exists a function $v \in \mathcal{C}(\Omega)$ such that $u = v\lambda$ -a.e.*

For every open subset U of $\mathbb{R}^n$ let $\mathcal{H}^+(U)$, resp. $\mathcal{S}^+(U)$ denote the set of all positive harmonic functions on U , resp. superharmonic functions on U .

Proof of Theorem 1. Let $\nu = \lambda + |\mu|$ and fix $x \in \Omega$. By [BHH, p. 132], there exists $r > 0$ such that the closure of the ball $B := \{y \in \mathbb{R}^n : \|y - x\| < r\}$ is contained in Ω and $G_B^{s|\mu|} \leq \frac{1}{3}s$ for all $s \in \mathcal{S}^+(B)$. Given a Borel measurable function f on B , let

$$Pf(y) := \int G_B(y, z)|f(z)|\mu(dz) \quad (y \in B)$$

and

$$Kf(y) := \int G_B(y, z)f(z)\mu(dz) \quad (y \in B, Pf(y) < \infty).$$

Since u is $|\mu|$ -integrable, the function $p := Pu$ is superharmonic on B (it is a potential). Hence the set $\{p = \infty\}$ is polar, $\nu(\{p = \infty\}) = 0$ (see, e.g., [BH, pp. 288–289]). Let h_0 denote the harmonic function on Ω satisfying $u + G_\Omega^{u\mu} = h_0$ ν -a.e. Then

$$N := \{y \in B : p(y) = \infty \text{ or } u(y) + G_\Omega^{u\mu} \neq h_0(y)\}$$

is a ν -null set. Denoting by h'_1, h'_2 the harmonic part of the restriction of the superharmonic function $G_\Omega^{(u\mu)^+}, G_\Omega^{(u\mu)^-}$, respectively, on B , we have

$$G_\Omega^{u\mu} = Ku + h'_1 - h'_2 \quad \text{on } B \setminus N.$$

Let $\alpha = \sup |h_0|(B)$, $h_1 = h'_1 + \alpha - h_0|_B$, $h_2 = h'_2 + \alpha$, and $h = h_2 - h_1$. Then $h_1, h_2 \in \mathcal{H}^+(B)$ and

$$u + Ku = h \quad \text{on } B \setminus N.$$

Since $K1_N = 0$, an immediate induction yields that for every natural m

$$\begin{aligned} |K^m h| &\leq P^m(h_1 + h_2) \leq 3^{-m}(h_1 + h_2), \\ |K^{m+1} u| &\leq P^{m+1} u \leq 3^{-m} p \quad \text{on } B \setminus N, \end{aligned}$$

and

$$u = \sum_{i=0}^m (-K)^i h + (-K)^{m+1} u \quad \text{on } B \setminus N.$$

Thus the proof is finished taking $v = \sum_{i=0}^\infty (-K)^i h$ if we can show that each function $K^i h$ is continuous on B . To that end fix i and a ball B' with $\overline{B'} \subset B$, and let $g = K^i h$, $g' = 1_{B'} g$. Since μ is a Kato measure and g is bounded on B' , we know that Kg' is continuous on B . Moreover, $Kg - Kg' = K(1_{B \setminus B'} g)$ is harmonic on B' . Thus $K^{i+1} h = Kg$ is continuous on B' .

By [BHH, p. 132], the continuous real solutions of $Lu = 0$ yield a harmonic space $(\Omega, {}^L\mathcal{H})$. In order to distinguish the corresponding classes of functions from the classical ones, we shall use the prefix “ L –”. In particular, an L -harmonic function g on U is simply a function $g \in \mathcal{C}(U)$ satisfying $Lg = 0$ on U . The following result is an immediate consequence of [BHH].

Theorem 2. *Let F be a polar subset of Ω that is closed in Ω , let Ω' be an open subset of Ω admitting a continuous Green function ${}^LG_{\Omega'}$ for L , and let $u \geq 0$ be an L -harmonic function on $\Omega' \setminus F$. Then there exists a unique measure κ on $\Omega' \cap F$ and a unique L -harmonic function $g \geq 0$ on Ω' satisfying $u = g + ({}^LG_{\Omega'})^\kappa$ on $\Omega' \setminus F$.*

Proof. Extend u to a function on Ω' defining

$$u(z) := \liminf_{x \rightarrow z} u(x) \quad (z \in \Omega' \cap F).$$

By [CC, p. 145] (or [BH, p. 380]), u is L -superharmonic on Ω' . Let g be the greatest L -harmonic minorant of u in Ω' . Then $p := u - g$ is an L -potential on Ω' . Hence there exists a unique measure κ on Ω' such that $p = ({}^LG_{\Omega'})^\kappa$ (see [B, p. 252]). Since p is L -harmonic on $\Omega' \setminus F$, we have $\kappa(\Omega' \setminus F) = 0$. If $\tilde{g}$ is L -harmonic on Ω' such that $u = \tilde{g} + ({}^LG_{\Omega'})^{\tilde{\kappa}}$ on $\Omega' \setminus F$, then $g + ({}^LG_{\Omega'})^\kappa = \tilde{g} + ({}^LG_{\Omega'})^{\tilde{\kappa}}$ on Ω' since F is (L) -polar. Thus $g = \tilde{g}$, $\kappa = \tilde{\kappa}$ by the Riesz representation theorem for $(\Omega', {}^L\mathcal{H})$.

Remarks. 1. Since $|\mu|(F) = 0$, there always exists an open neighborhood Ω' of F in Ω satisfying $\|G_{\Omega'}^{1, |\mu|}\|_\infty < 1$ and then a continuous Green function ${}^LG_{\Omega'}$ exists by [BHH, pp. 107, 116–118].

2. Note that our assumption allows that F has accumulation points in the boundary of Ω .

3. If $\limsup_{x \rightarrow z} u(x) < \infty$ for every $z \in F$ then it follows already from [CC, p. 146] that u can be extended to an L -harmonic function g on Ω (and $\kappa = 0$).

4. If F is a singleton $\{x_0\}$ then it suffices to know that $u(x) = o(\|x - x_0\|^{2-n})$ as x tends to x_0 in order to conclude that $v = u$ (see [BHH, p. 132]).

In the following let ρ be a Kato measure on Ω .

Proposition. *Let B be an open ball such that $\overline{B} \subset \Omega$ and $G_B^{s, |\mu|} \leq \frac{1}{3}s$ for every $s \in \mathcal{S}^+(B)$. Let F be a polar subset of B that is closed in B , and let $u \in \mathcal{C}(B \setminus F)$ such that $Lu = \rho$ on $B \setminus F$ and $u \geq -s|_{B \setminus F}$ for some $s \in \mathcal{S}^+(B)$. Then there exists a unique signed measure κ on F and a unique function $v \in \mathcal{C}(B)$ satisfying $Lv = \rho$ on B and $u = v + ({}^LG_B)^\kappa$ on $B \setminus F$.*

Proof. Let $\mathcal{B}_b(B)$, $\mathcal{C}_b(B)$ denote the space of all bounded Borel functions, bounded continuous functions, respectively, on B . Defining an integral operator K as in the proof of Theorem 1, we have $K(\mathcal{B}_b(B)) \subset \mathcal{C}_b(B)$, $\|K\| \leq \frac{1}{3}$. Since $q := G_B^{-1, \rho} \in \mathcal{C}_b(B)$, we obtain that

$$u_0 = \sum_{m=0}^{\infty} (-K)^m q \in \mathcal{C}_b(B).$$

Clearly $u_0 + Ku_0 = q$, hence $Lu_0 = \frac{1}{2}\Delta q = \rho$ on B . So we may assume without loss of generality that $\rho = 0$.

By [BHH, p. 113], we have a Green function ${}^L G_B$ satisfying

$$\frac{1}{2} G_B \leq {}^L G_B \leq \frac{3}{2} G_B.$$

There exists $h \in \mathcal{H}^+(B)$ and a measure σ on B such that $s = h + G_B^\sigma$. Let

$$g = \sum_{m=0}^{\infty} (-K)^m h, \quad p = {}^L G_B^{1_F \sigma}, \quad q = {}^L G_B^{1_{B \setminus F} \sigma}.$$

Then g is L -harmonic on B , p is L -harmonic on $B \setminus F$, and $q|_{B \setminus F} = {}^L G_{B \setminus F}^{1_{B \setminus F} \sigma}$ is an L -potential on $B \setminus F$. Since

$$u + 2(g + p + q) \geq u + s \geq 0 \quad \text{on } B \setminus F,$$

the general minimum principle for harmonic spaces yields that the L -harmonic function $u + 2(g + p)$ on $B \setminus F$ is positive. So by Theorem 2 there exists a unique measure τ on F and a unique L -harmonic function w on B such that $u + 2(g + p) = w + ({}^L G_B)^\tau$ on $B \setminus F$. Taking $v = w - 2g$ and $\kappa = \tau - 2 \cdot 1_F \sigma$, we thus have $u = v + ({}^L G_B)^\kappa$ on $B \setminus F$. The uniqueness of τ , w implies the uniqueness of κ , v .

A straightforward covering argument leads to the following

Corollary. *Let F be a compact polar subset of Ω and let $u \in \mathcal{E}(\Omega \setminus F)$ such that $Lu = \rho$ on $\Omega \setminus F$ and $u \geq -s|_{\Omega \setminus F}$ for some $s \in \mathcal{S}^+(\Omega)$. Then there exists a unique measure κ on F such that for every sufficiently small neighborhood V of F in Ω there exists a unique function $v \in \mathcal{E}(V)$ satisfying $Lv = \rho$ on V and $u = v + ({}^L G_V)^\kappa$ on $V \setminus F$.*

REFERENCES

- [AS] M. Aizenman and B. Simon, *Brownian motion and Harnack inequality for Schrödinger operators*, Comm. Pure Appl. Math. **35** (1982), 209–273.
- [BH] J. Bliedtner and W. Hansen, *Potential theory—an analytic and probabilistic approach to Balayage*, Universitext, Springer, Berlin and New York, 1986.
- [B] A. Boukricha, *Das Picard-Prinzip und verwandte Fragen bei Störung von harmonischen Räumen*, Math. Ann. **239** (1979), 247–270.
- [BHH] A. Boukricha, W. Hansen, and H. Hueber, *Continuous solutions of the generalized Schrödinger equation and perturbation of harmonic spaces*, Exposition. Math. **5** (1987), 97–135.
- [CC] C. Constantinescu and C. Cornea, *Potential theory on harmonic spaces*, Grundlehren Math. Wiss., vol. 158, Springer, Berlin and New York, 1972.
- [H] W. Hansen, *Valeurs propres pour l'opérateur de Schrödinger*, Séminaire de Théorie de Potentiel, Paris, No. 9, Lecture Notes in Math., vol. 1393, Springer, Berlin and New York, 1989, pp. 117–134.
- [VY] J. L. Vazquez and C. Yahur, *Isolated singularities of the Schrödinger equation with a good potential*, Trans. Amer. Math. Soc. **315** (1989), 711–720.

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