UNCONDITIONAL CONVERGENCE AND WEAKLY COMPACT MAPS

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We characterize the barrelled spaces E such that every linear continuous map from E to l_1 carries bounded sets into relatively compact sets. These characterizations involve unconditional convergence of series, Edgar's ordering for Banach spaces, perfect sequence spaces, copies of c_0 , and Orlicz-Pettis topologies.

1. INTRODUCTION

A Banach space E is said to be Grothendieck if weak* and weak sequential convergence in E' coincide. Grothendieck spaces are close to the heart of every vector measure theorist (see [2]). Here is a list of reformulations of the definition of Grothendieck spaces which can be also found in [2, p. 197].

Theorem 1 [2]. *Any one of the following statements about a Banach space E implies all the others:*

- (i) E is a Grothendieck space.
- (ii) Every bounded linear operator from E to c_0 is weakly compact.
- (iii) For all Banach spaces F such that F' has a weak* sequentially compact unit ball, every bounded linear operator from E to F is weakly compact.
- (iv) For all weakly compactly generated Banach spaces F , every bounded linear operator from E to F is weakly compact.

These kinds of results respond to a widely treated question in the literature (see [2, 3]): To find conditions on a space E such that every linear continuous map from E to F (or from F to E) is weakly compact, where F is a "well-known" space such as L^p , l^p , $C(K)$, c_0 , and so on.

In the framework of locally convex spaces, we are going to show that if we change "sequential convergence" for "unconditional convergence of series" in the definition of Grothendieck spaces, we can obtain a similar result to Theorem 1 with the Banach space l_1 and some other classes of Banach spaces. Moreover, we also give some other characterizations that involve perfect sequence spaces, copies of c_0 , and Orlicz-Pettis topologies.

Received by the editors November 12, 1990 and, in revised form, June 10, 1991.

1991 *Mathematics Subject Classification*. Primary 47B07, 46B20; Secondary 46A08.

In what follows, by space we mean Hausdorff locally convex space. Although we refer the reader for the terminology and the notation to Jarchow's monograph [6], we shall recall a few definitions.

A series $\sum x_n$ in a space is said to be λ -multiplier convergent (Cauchy) for λ a sequence space if $\sum \alpha_n x_n$ converges (is Cauchy) for all $(\alpha_n) \in \lambda$. We usually write bounded multiplier instead of l_∞ -multiplier and subseries convergent (Cauchy) instead of m_0 -multiplier convergent (Cauchy), where m_0 is the space of sequences with finite range.

It is worth mentioning the fact that “unconditionally Cauchy”, “subseries Cauchy”, and “bounded multiplier Cauchy” are equivalent concepts (see [10]). This will be widely used in the sequel.

For a series $\sum x_n$, the set $\Omega = \{\sum_{i \in \sigma} x_i : \sigma \text{ a finite subset of } \mathbb{N}\}$ is called its range. Ω is bounded if and only if $\sum x_n$ is c_0 -multiplier Cauchy (see [10]).

Finally, we denote by $\mathcal{J}$ the canonical mapping from a space into its bidual.

2. ORLICZ-PETTIS TOPOLOGIES

The Orlicz-Pettis theorem asserts that every subseries $\sigma(E, E')$ -convergent series in a space E is subseries convergent under any compatible topology of the dual pair (E, E') . In fact, even more is known. Twedde [11] proved that on every space $[E, \tau]$, there exists a finest locally convex Hausdorff topology that has the same subseries convergent series as the original one. This topology is denoted by $\mathcal{OP}(\tau)$ and is called the Orlicz-Pettis topology associated to τ . Now, the Orlicz-Pettis theorem appears in the form

$$\mathcal{OP}(\mu(E, E')) = \mathcal{OP}(\sigma(E, E')).$$

Given two topologies $\tau_1 \leq \tau_2$ on a space E , the maximal property of the Orlicz-Pettis topologies clearly implies that $\mathcal{OP}(\tau_1) \leq \mathcal{OP}(\tau_2)$.

Twedde also characterized $\mathcal{OP}$ -topologies in the following way.

Lemma 1 [11]. $\mathcal{OP}(\tau)$ is the Mackey topology $\mu(E, E^\Sigma)$, where E^Σ is the subspace of all linear forms y on E such that for each subseries τ -convergent series $\sum x_n$ we have

$$\left\langle \sum_{n=1}^{\infty} x_n, y \right\rangle = \sum_{n=1}^{\infty} \langle x_n, y \rangle.$$

First of all, we are going to study how $\mathcal{OP}$ -topologies deal with continuity of linear mappings.

Lemma 2. Let $[E, \tau_E], [F, \tau_F]$ be spaces. Then, every linear continuous map from E to F is also $\mathcal{OP}(\tau_E)$ - $\mathcal{OP}(\tau_F)$ continuous.

Proof. Let T be a linear continuous map from E to F . By Lemma 1, $\mathcal{OP}$ -topologies are Mackey topologies, so it is sufficient to show that $T^*(F^\Sigma) \subset E^\Sigma$. But, given $z \in F^\Sigma$ and a subseries τ_E -convergent series $\sum x_n$, we have

$$\left\langle \sum_{n=1}^{\infty} x_n, T^*(z) \right\rangle = \left\langle T \left(\sum_{n=1}^{\infty} x_n \right), z \right\rangle = \left\langle \sum_{n=1}^{\infty} T(x_n), z \right\rangle = \sum_{n=1}^{\infty} \langle x_n, T^*(z) \rangle. \quad \square$$

The rest of this section is devoted to find topologies τ such that $\mathcal{OP}(\tau) = \tau$. We give two examples, one in sequence spaces and the other in Banach spaces.

Given a sequence space λ , we can assign it another sequence space $\lambda^\times$, its α -dual. $\lambda^\times$ is defined to be the set of all sequences $x = (x_n)$ for which the series $\sum x_n y_n$ converges absolutely, for all $y = (y_n) \in \lambda$. If λ contains ϕ , the space of all sequences with only finitely many nonzero coordinates, then $(\lambda, \lambda^\times)$ is a dual pair. A sequence space λ is said to be normal if every sequence (y_n) with $|y_n| \leq |x_n|$, $n = 1, 2, \dots$, for some $(x_n) \in \lambda$, also belongs to λ . Clearly, perfect sequence spaces ($\lambda = \lambda^{\times \times}$) are normal spaces. For more details see [8, Chapter 30].

Lemma 3. *If λ is a normal sequence space and contains ϕ , then $\mathcal{OP}(\mu(\lambda, \lambda^\times)) = \mu(\lambda, \lambda^\times)$.*

Proof. If we show that we can identify $\lambda^\times$ with $\lambda_{\mu(\lambda, \lambda^\times)}^\Sigma$, then the result follows from Lemma 1.

On the one hand, every $z = (z_n) \in \lambda^\times$ defines a linear $\sigma(\lambda, \lambda^\times)$ -continuous functional on λ , namely, $f_z(x) = \langle x, z \rangle = \sum_{n=1}^\infty x_n z_n$. So f_z maps subseries $\mu(\lambda, \lambda^\times)$ -convergent series into subseries convergent series and then $f_z \in \lambda_{\mu(\lambda, \lambda^\times)}^\Sigma$.

On the other hand, let be $z \in \lambda_{\mu(\lambda, \lambda^\times)}^\Sigma$. For every $x = (x_n) \in \lambda$, we have that the series $\sum x_n e_n$ is absolutely $\sigma(\lambda, \lambda^\times)$ -convergent to x , where e_n denotes the sequence all of whose coordinates vanishes except the n th, which is equal to one. Therefore

$$\begin{aligned} \langle z, x \rangle &= \left\langle z, \sum_{n=1}^\infty x_n e_n \right\rangle = \sum_{n=1}^\infty x_n \langle z, e_n \rangle \\ &= \langle x, ((\langle z, e_n \rangle)_n)_{(\lambda, \lambda^\times)} \rangle = f_{((\langle z, e_n \rangle)_n)}(x). \quad \square \end{aligned}$$

Edgar [4] introduced the following order relation (reflexive and transitive) on the class of all Banach spaces:

$$E < F \equiv \mathcal{J}(E) = \bigcap_{T \in L(E, F)} (T^{**})^{-1}(\mathcal{J}(F)),$$

where $L(E, F)$ denotes as usual the set of linear continuous maps from E to F .

We are specially interested on Banach spaces E such that $E < l_1$. Edgar proved that this class can be characterized by the so-called property (X):

For every $y \in E''$, one has $y \in \mathcal{J}(E)$ if and only if for every unconditionally $\sigma(E', E)$ -convergent series $\sum_{n \geq 1} f_n$, $\langle y, \sum_{n=1}^\infty f_n \rangle = \sum_{n=1}^\infty \langle y, f_n \rangle$.

This property has connections with unique preduals (see [5]). Godefroy and Talagrand [5] show that the following Banach spaces have property (X):

- (1) Separable weakly sequentially complete Banach lattices (*).
- (2) Separable w.s.c. spaces that are complemented into a Banach lattice.
- (3) Separable spaces with local unconditional structure that do not contain l_n^∞ uniformly.
- (4) Preduals of von Neumann algebras.
- (5) Dual Banach spaces of separable Banach spaces with no copy of l_1 and with the property (u) of Pelczynsky.

Our next result on $\mathcal{OP}$ -topologies is about the above class $E < l_1$.

Lemma 4. *Let E be a Banach space. Then,*

- (i) $E < l_1$ if and only if $\mathcal{J}(E) = E'' \cap (E')_{\mu(E', E)}^\Sigma$;
- (ii) If $E < l_1$ then $\mathcal{OP}(\mu(E', E)) = \mu(E', E)$.

Proof. (i) This is a consequence of the definition of property (X) and Lemma 1, keeping in mind that for every Banach space E the weak* subseries convergent series and the weak* unconditionally Cauchy series in E' coincide.

(ii) Let $[F, \tau = \mu(F, F')]$ be a metrizable complete space. By Lemma 1, if we show that $F' = F^\Sigma$ then $\mathcal{OP}(\tau) = \tau$.

It is enough to show that $F^\Sigma \subset F'$. Let $f \in F^\Sigma$ be such that f is not τ -continuous. Since F is metrizable, f is not sequentially τ -continuous and therefore, there exists a τ -null sequence (x_n) in E such that $f(x_n) \geq 1$ for all $n \in \mathbb{N}$. We can also extract a subsequence (x_{n_k}) such that $\|x_{n_k}\| \leq 1/k^2$. But, E is complete so

$$\left\langle \sum_{k=1}^{\infty} x_{n_k}, f \right\rangle = \sum_{k=1}^{\infty} \langle x_{n_k}, f \rangle.$$

Finally, the convergence of the last series implies that $f(x_{n_k})_k$ tends to zero, which is impossible.

Now, we are able to prove statement (ii). Since E is a Banach space, E' endowed with the strong topology $\beta(E', E)$ is again a Banach space (metrizable and complete), so the following chain of inclusions is having

$$\mathcal{OP}(\mu(E', E)) \leq \mathcal{OP}(\beta(E', E)) = \beta(E', E).$$

This means that $(E')^\Sigma \subset E''$, so by (i), $\mathcal{J}(E) = E'' \cap (E')^\Sigma = (E')^\Sigma$ and by Lemma 1,

$$\mathcal{OP}(\mu(E', E)) = \mu(E', (E')^\Sigma) = \mu(E', E). \quad \square$$

3. THE MAIN THEOREM

We are now ready to prove the theorem that gives the characterization mentioned in the abstract.

Theorem 2. *Let E be a barrelled space. Then, the following conditions are equivalent:*

- (i) $\mathcal{OP}(\sigma(E', E))$ is finer than $\beta(E', E)$.
- (ii) $[E', \beta(E', E)]$ has no copy of c_0 .
- (iii) Weak* and weak unconditional convergence of series in E' coincide.
- (iv) Every linear continuous map from E to l_1 carries bounded sets into relatively compact sets.
- (v) For all Banach spaces F such that $F < l_1$ in Edgar's ordering, every linear continuous map from E to F carries bounded sets into weakly relatively compact sets.
- (vi) For all separable weakly sequentially complete Banach lattices F , every linear continuous map from E to F carries bounded sets into weakly relatively compact sets.
- (vii) For all perfect sequence spaces λ , every linear continuous map from E to $[\lambda, \mu(\lambda, \lambda^\times)]$ maps bounded sets into weakly relatively compact sets.

Proof. Using (*), one has (v) $\Rightarrow$ (vi). Since l_1 is a weakly sequentially complete Banach space with basis, (vi) $\Rightarrow$ (iv), and since l_1 is a perfect sequence space (see [8, p. 406]), we have (vii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (iii). Let $\sum x_n$ be an unconditionally convergent series in $[E', \sigma(E', E)]$. Since E is a barrelled space, E' is weak* sequentially complete, so the series $\sum x_n$ is bounded multiplier $\sigma(E', E)$ -convergent. This allows us to define the linear continuous mapping

$$T: E \rightarrow l_1, \quad x \mapsto T(x) = (\langle x, x_n \rangle)_n.$$

By (iv) and the characterization of relatively compact sets of l_1 (see [8, p. 282]), we have that for every bounded set A in E ,

$$\sup_{a \in A} \left| \left\langle a, \sum_{k=n}^m x_{r_k} \right\rangle \right| \leq \sup_{a \in A} \sum_{k=n}^{\infty} |\langle a, x_k \rangle| \xrightarrow{n} 0.$$

This means that the series $\sum x_n$ is unconditionally $\beta(E', E)$ -Cauchy and, by the Bourbaki-Robertson Lemma, $\sum x_n$ is also $\beta(E', E)$ -convergent, therefore $\sigma(E', E'')$ -convergent.

(iii) $\Rightarrow$ (ii). Suppose that we have an isomorphism T from c_0 to a subspace of $[E', \beta(E', E)]$. Since $\sum e_n$ is weakly unconditionally Cauchy in c_0 and E' is weak* sequentially complete, $\sum T(e_n)$ is bounded multiplier convergent in $[E', \sigma(E', E)]$. But, by (iii) and the Orlicz-Pettis theorem, we obtain that $\sum T(e_n)$ is also bounded multiplier $\beta(E', E)$ -convergent.

Finally, T is bicontinuous onto its image, so $(\sum_{k=1}^n e_k)_n$ must be a norm Cauchy sequence in c_0 , which is absurd.

(ii) $\Rightarrow$ (i). By the Bourbaki-Robertson Lemma, we only need to show that each subseries $\sigma(E', E)$ -convergent series in E' is subseries $\beta(E', E)$ -Cauchy.

Assume that this is not true. This implies that there exists a subseries $\sigma(E', E)$ -convergent series $\sum x_n$, a $\beta(E', E)$ -neighbourhood U of 0, and a sequence of finite subsets α_n of positive integers with $\sup \alpha_n < \inf \alpha_{n+1}$ that verify

$$z_n := \sum_{i \in \alpha_n} x_i \notin U \quad \text{for all } n \in \mathbb{N}.$$

The range of $\sum x_n$ is $\sigma(E', E)$ -bounded, so the range of $\sum z_n$ is $\sigma(E', E)$ -bounded. The range of the series $\sum z_n$, however, is also $\beta(E', E)$ -bounded by the Banach-Mackey theorem, and we conclude that $\sum z_n$ is c_0 -multiplier $\beta(E', E)$ -Cauchy.

But, since E' is weak* sequentially complete, $\sum x_n$ is actually bounded multiplier $\sigma(E', E)$ -convergent, so $\sum z_n$ is c_0 -multiplier $\sigma(E', E)$ -convergent, and by the Bourbaki-Robertson Lemma c_0 -multiplier $\beta(E', E)$ -convergent. This allows us to define the following linear continuous mapping

$$T: c_0 \longrightarrow [E', \beta(E', E)], \quad (a_n) \mapsto T(a_n) = \sum_{n=1}^{\infty} a_n z_n.$$

By (ii) and [7], we obtain that $T(e_n)$ must tend to zero, so (z_n) is a $\beta(E', E)$ -null sequence and we get a contradiction.

(i) $\Rightarrow$ (vii). Let T be a map from E to λ in the hypothesis of (vii). Since T is weakly continuous and $\lambda = \lambda^{\times \times}$, the adjoint map T^* is $\sigma(\lambda^{\times}, \lambda^{\times \times})$ - $\sigma(E', E)$ -continuous.

Now, if we consider the associated Orlicz-Pettis topologies, we have that T^* is also $\mathcal{OP}(\sigma(\lambda^{\times}, \lambda^{\times \times}))$ - $\mathcal{OP}(\sigma(E', E))$ -continuous by Lemma 2. But by Lemma 3 and (i), T^* is $\mu(\lambda^{\times}, \lambda^{\times \times})$ - $\beta(E', E)$ -continuous.

Again, since $\lambda = \lambda^{\times \times}$, we have that T^* is $\sigma(\lambda^\times, \lambda)$ - $\sigma(E', E'')$ -continuous. This implies that $T^{**}(E'') \subset \mathcal{J}(\lambda)$. Finally by [9, p. 204], we deduce that T maps bounded sets into weakly relatively compact sets.

(i) $\Rightarrow$ (v). Let T be a map from E to F in the hypothesis of (v). Since T is weakly continuous, its adjoint map T^* is weakly* continuous.

Now, if we consider the associated Orlicz-Pettis topologies, we have that T^* is $\mathcal{OP}(\sigma(F', F))$ - $\mathcal{OP}(\sigma(E', E))$ -continuous by Lemma 2. And by Lemma 4 and (i), T^* is also $\mu(F', F)$ - $\beta(E', E)$ -continuous. Therefore, T^* is $\sigma(F', F)$ - $\sigma(E', E'')$ -continuous. This implies that $T^{**}(E'') \subset \mathcal{J}(F)$.

Finally by [9, p. 204], we obtain that T maps bounded sets into weakly relatively compact sets. $\square$

Remarks. (1) We notice that if E is a normed space, the linear continuous maps that carry bounded sets into weakly relatively compact sets are exactly the weakly compact maps.

(2) Clearly, the hypothesis of barrelledness can be weakened. In fact, Theorem 2 is even true for a space E such that $[E', \sigma(E', E)]$ is sequentially complete.

(3) In the statement (vi), separable weakly sequentially complete Banach lattices can be replaced by other classes of Banach spaces having property (X) and that contain l_1 .

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