ASYMPTOTIC BEHAVIOR OF ALMOST-ORBITS OF NONLINEAR SEMIGROUPS OF NON-LIPSCHITZIAN MAPPINGS IN HILBERT SPACES

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ABSTRACT. Let C be a nonempty closed convex subset of a Hilbert space H , $\mathcal{F} = \{T(t): t \geq 0\}$ be a continuous nonlinear asymptotically nonexpansive semigroup acting on C with a nonempty fixed point set $F(\mathcal{F})$, and $u: [0, \infty) \rightarrow C$ be an almost-orbit of $\mathcal{F}$. Then $\{u(t)\}$ almost converges weakly to a fixed point of $\mathcal{F}$, i.e., there exists an element y in $F(\mathcal{F})$ such that

$$\text{weak-lim } \frac{1}{t} \int_0^t u(r+h) dr = y \quad \text{uniformly for } h \geq 0.$$

This implies that $\{u(t)\}$ converges weakly to a fixed point of $\mathcal{F}$ if and only if $\{u(t+h) - u(t)\}$ converges weakly to zero as t tends to infinity for each $h \geq 0$.

1. INTRODUCTION

Let C be a nonempty closed convex subset of a Hilbert space H and let $\mathcal{F} = \{T(t): t \geq 0\}$ be a (one-parameter) semigroup acting on C , i.e., each $T(t)$ is a self-mapping of C and for each x in C and $t, s \geq 0$ we have

- (i) $T(0)x = x$ and
- (ii) $T(s+t)x = T(s)T(t)x$.

$\mathcal{F}$ is said to be continuous if for each x in C , the mapping $t \rightarrow T(t)x$ is continuous. Recall that a semigroup $\mathcal{F}$ on C is said to be

- (a) nonexpansive if $\|T(t)x - T(t)y\| \leq \|x - y\|$ for x, y in C and $t \geq 0$;
- (b) uniformly asymptotically nonexpansive [6] if there is a function $b: [0, \infty) \rightarrow [0, \infty)$ with $\limsup_{t \rightarrow \infty} b(t) \leq 1$ such that $\|T(t)x - T(t)y\| \leq b(t)\|x - y\|$ for x, y in C and $t \geq 0$;
- (c) asymptotically nonexpansive [6] if for each x in C ,

$$\limsup_{t \rightarrow \infty} \left\{ \sup_{y \in C} [\|T(t)x - T(t)y\| - \|x - y\|] \right\} \leq 0.$$

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It is easily seen that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) and that both the inclusions are proper (cf. [6, p. 112]).

Recently, Miyadera and Kobayasi [9] introduced the notion of almost-orbits of nonexpansive semigroups on C and proved the weak almost convergence of such an almost-orbit in a uniformly convex Banach space with a Fréchet differentiable norm. Takahashi and Zhang [10] extended this notion to uniformly asymptotically nonexpansive semigroups in Hilbert spaces.

In this paper, we introduce the concept of almost-orbits for asymptotically nonexpansive semigroups. We shall prove that if $u: [0, \infty) \rightarrow C$ is an almost-orbit of an asymptotically nonexpansive semigroup $\mathcal{F} = \{T(t): t \geq 0\}$ such that each $T(t)$ is continuous and the fixed point set $F(\mathcal{F})$ of $\mathcal{F}$ is nonempty, then it almost converges weakly to some y in $F(\mathcal{F})$, i.e.,

$$\text{weak-lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t u(r+h) dr = y$$

uniformly for $h \geq 0$. This extends Theorem 4.1 of Miyadera and Kobayasi [9] to semigroups of non-Lipschitzian mappings in Hilbert spaces and implies that $\{u(t)\}$ converges weakly to a member of $F(\mathcal{F})$ if and only if u is weakly asymptotically regular, that is, $u(t+h) - u(t) \rightarrow 0$ weakly as $t \rightarrow \infty$ for each $h \geq 0$. We also prove the existence of a nonexpansive retraction from the set of almost-orbits of the semigroup $\mathcal{F}$ onto the fixed point set $F(\mathcal{F})$ of $\mathcal{F}$. Related results to our paper can be found in Bruck [2], Kirk [5], Lau [7], Takahashi and Zhang [10], and You and Xu [11].

2. PRELIMINARIES

Throughout this section, let C be a nonempty closed convex subset of a Hilbert space H and $\mathcal{F} = \{T(t): t \geq 0\}$ be an asymptotically nonexpansive semigroup acting on C . We shall first introduce the notion of almost-orbit of $\mathcal{F}$ and prove some lemmas that will be used later.

Definition. A continuous function $u: [0, \infty) \rightarrow C$ is said to be an almost-orbit of $\mathcal{F}$ if

$$\lim_{t \rightarrow \infty} \left(\sup_{s \geq 0} \|u(s+t) - T(s)u(t)\| \right) = 0.$$

Clearly, for each x in C , the orbit $\{T(t)x: t \geq 0\}$ of $\mathcal{F}$ at x is an almost-orbit of $\mathcal{F}$ provided $\mathcal{F}$ is continuous.

Denote by $F(\mathcal{F})$ the fixed point set of $\mathcal{F}$, i.e., $F(\mathcal{F}) = \{x \in C: T(t)x = x \text{ for all } t \geq 0\}$.

Lemma 2.1. *If each $T(t)$ is continuous then $F(\mathcal{F})$ is closed and convex.*

Proof. It needs only to prove the convexity of $F(\mathcal{F})$. Suppose $f, g \in F(\mathcal{F})$ and let $p = \frac{1}{2}(f+g)$. For an arbitrary $\varepsilon > 0$, since $\mathcal{F}$ is asymptotically nonexpansive, there exists a $t_0 > 0$ such that for $t \geq t_0$,

$$\sup_{y \in C} [\|q - T(t)y\| - \|q - y\|] < \varepsilon \quad \text{for } q = f, g,$$

and hence for $t \geq t_0$,

$$\begin{aligned} \|T(t)p - p\|^2 &= \frac{1}{2}\|T(t)p - f\|^2 + \frac{1}{2}\|T(t)p - g\|^2 - \frac{1}{4}\|f - g\|^2 \\ &\leq (\tfrac{1}{2}\|f - g\| + \varepsilon)^2 - \tfrac{1}{4}\|f - g\|^2. \end{aligned}$$

It follows that $\lim_{t \rightarrow \infty} \|T(t)p - p\| = 0$, i.e., $T(t)p \rightarrow p$ strongly. By continuity of $T(t)$, p is a fixed point of each $T(t)$ and, therefore, p is in $F(\mathcal{F})$. $\square$

Lemma 2.2. *If u and v are almost-orbits of $\mathcal{F}$ then $\lim_{t \rightarrow \infty} \|u(t) - v(t)\|$ exists. In particular, for every $f \in F(\mathcal{F})$, $\lim_{t \rightarrow \infty} \|u(t) - f\|$ exists.*

Proof. Set

$$\varphi(t) = \sup_{s \geq 0} \|u(s+t) - T(s)u(t)\|$$

and

$$\psi(t) = \sup_{s \geq 0} \|v(s+t) - T(s)v(t)\|.$$

Then $\lim_{t \rightarrow \infty} \varphi(t) = \lim_{t \rightarrow \infty} \psi(t) = 0$. Since $\mathcal{F}$ is asymptotically nonexpansive, for a fixed $t > 0$, one can find a s_0 (depending upon t) such that for $s \geq s_0$

$$\sup_{y \in C} (\|T(s)u(t) - T(s)y\| - \|u(t) - y\|) < \varepsilon;$$

in particular,

$$\|T(s)u(t) - T(s)v(t)\| < \|u(t) - v(t)\| + \varepsilon.$$

Therefore for $s \geq s_0$,

$$\begin{aligned} \|u(t+s) - v(t+s)\| &\leq \|u(t+s) - T(s)u(t)\| \\ &\quad + \|T(s)u(t) - T(s)v(t)\| + \|v(t+s) - T(s)v(t)\| \\ &\leq \varphi(t) + \psi(t) + \|u(t) - v(t)\| + \varepsilon, \end{aligned}$$

and thus for each fixed $t > 0$,

$$\begin{aligned} \limsup_{s \rightarrow \infty} \|u(s) - v(s)\| &= \limsup_{s \rightarrow \infty} \|u(t+s) - v(t+s)\| \\ &\leq \varphi(t) + \psi(t) + \|u(t) - v(t)\|. \end{aligned}$$

It follows that $\limsup_{t \rightarrow \infty} \|u(t) - v(t)\| \leq \liminf_{t \rightarrow \infty} \|u(t) - v(t)\|$, completing the proof. $\square$

By Lemma 2.1, if $F(\mathcal{F})$ is nonempty, the metric projection P from H onto $F(\mathcal{F})$ is well defined. We now prove

Lemma 2.3. *Suppose the fixed point set $F(\mathcal{F})$ of $\mathcal{F}$ is nonempty. Then for each almost-orbit u of $\mathcal{F}$, $\{Pu(t)\}$ converges strongly to some z in $F(\mathcal{F})$. Moreover, if $\{u(t)\}$ converges weakly to a point y in $F(\mathcal{F})$, then $y = z$.*

Proof. As before, we set $\varphi(t) = \sup_{s \geq 0} \|u(s+t) - T(s)u(t)\|$. First, we show that $\lim_{t \rightarrow \infty} \|Pu(t) - u(t)\|$ exists. In fact, for an arbitrary $\varepsilon > 0$ and a fixed $t > 0$, we can choose an s_0 so large that for $s \geq s_0$

$$\sup_{y \in C} (\|Pu(t) - T(s)y\| - \|Pu(t) - y\|) < \varepsilon.$$

It then follows that for $s \geq s_0$,

$$\begin{aligned} \|Pu(t+s) - u(t+s)\| &\leq \|Pu(t) - u(t+s)\| \\ &\leq \|Pu(t) - T(s)u(t)\| + \|T(s)u(t) - u(t+s)\| \\ &\leq \|Pu(t) - u(t)\| + \varphi(t) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this yields

$$\limsup_{s \rightarrow \infty} \|Pu(s) - u(s)\| \leq \liminf_{t \rightarrow \infty} \|Pu(t) - u(t)\|$$

so that $\lim_{t \rightarrow \infty} \|Pu(t) - u(t)\|$ exists. Now, since

$$\|Px - y\|^2 \leq \|x - y\|^2 - \|x - Px\|^2 \quad \text{for all } x \in H \text{ and } y \in F(\mathcal{F}),$$

by setting $x = u(t + s)$ and $y = Pu(t)$, we get for $s \geq s_0$

$$\begin{aligned} \|Pu(t + s) - Pu(t)\|^2 &\leq \|u(t + s) - Pu(t)\|^2 - \|u(t + s) - Pu(t + s)\|^2 \\ &\leq (\|Pu(t) - u(t)\| + \varphi(t) + \varepsilon)^2 - \|u(t + s) - Pu(t + s)\|^2, \end{aligned}$$

which implies that $\limsup_{t \rightarrow \infty} (\limsup_{s \rightarrow \infty} \|Pu(t) - Pu(s)\|) = 0$, which in turn implies (cf. [1, Lemma 3]) that $\{Pu(t)\}$ converges strongly to some point z in $F(\mathcal{F})$. Now suppose that $\{u(t)\}$ converges weakly to a y in $F(\mathcal{F})$. Then since $\operatorname{Re}\langle u(t) - Pu(t), f - Pu(t) \rangle \leq 0$ for all $f \in F(\mathcal{F})$, by taking the limit as $t \rightarrow \infty$, we get $\operatorname{Re}\langle y - z, f - z \rangle \leq 0$ for all $f \in F(\mathcal{F})$. Because $y \in F(\mathcal{F})$, we in particular obtain $\|y - z\|^2 = \operatorname{Re}\langle y - z, y - z \rangle \leq 0$ and thus $y = z$. The proof is complete. $\square$

By applying a result of Lim [8, Corollary 1], the proof of Lemma 5 in Takahashi and Zhang [10] can be modified to prove the following result and is thus omitted.

Lemma 2.4. *The limit z of $\{Pu(t)\}$ in Lemma 2.3 is the asymptotic center of $\{u(t)\}$ in $F(\mathcal{F})$, i.e., z is the unique point in $F(\mathcal{F})$ that minimizes the functional $\lim_{t \rightarrow \infty} \|u(t) - f\|$ over f in $F(\mathcal{F})$.*

We remark here that the original notion of asymptotic center was first introduced by Edelstein in [3] to obtain fixed point theorems in [3, 4].

3. ASYMPTOTIC BEHAVIOR

In this section, we prove the asymptotic behavior of an almost-orbit of an asymptotically nonexpansive semigroup in a Hilbert space. The main result, Theorem 3.2, extends Theorem 4.1 of Miyadera and Kobayasi [9] to semigroups of non-Lipschitzian mappings in Hilbert spaces.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and $\mathcal{F} = \{T(t): t \geq 0\}$ be a continuous asymptotically nonexpansive semigroup acting on C such that each $T(t)$ is continuous. Then the following two statements are equivalent:*

- (i) $F(\mathcal{F})$ is nonempty;
- (ii) there exists a bounded almost-orbit of $\mathcal{F}$.

Proof. (i) $\Rightarrow$ (ii). For each $x \in C$, $\{T(t)x: t \geq 0\}$ is an almost-orbit of $\mathcal{F}$. By Lemma 2.2, every almost-orbit of $\mathcal{F}$ is bounded.

(ii) $\Rightarrow$ (i). Let u be a bounded almost-orbit of $\mathcal{F}$. For each $t > 0$, set

$$S_t(u) = \frac{1}{t} \int_0^t u(r) dr, \quad \varphi(t) := \sup_{s \geq 0} \|u(s + t) - T(s)u(t)\|.$$

Then $\{S_t(u)\}$ is bounded and $\frac{1}{t} \int_0^t \varphi(r) dr \rightarrow 0$ as $t \rightarrow \infty$. Let f be a weak cluster point of $\{S_t(u)\}$. For an arbitrary $\varepsilon > 0$, we can choose a t_0 so large that $\|T(t)u(r) - T(t)f\|^2 \leq \|u(r) - f\|^2 + \varepsilon$ for all $t \geq t_0$ and $r \geq 0$ and that

$\varphi(t) < \varepsilon$ for all $t \geq t_0$. Thus for an arbitrary but fixed $s \geq t_0$, we have for $r \geq 0$ that

$$\begin{aligned} -\varepsilon &\leq \|u(r) - f\|^2 - \|T(s)u(r) - T(s)f\|^2 \\ &= \|u(r) - T(s)f\|^2 + \|T(s)f - f\|^2 + 2 \cdot \operatorname{Re}\langle u(r) - T(s)f, T(s)f - f \rangle \\ &\quad - \|T(s)u(r) - u(s+r)\|^2 - \|u(s+r) - T(s)f\|^2 \\ &\quad - 2 \cdot \operatorname{Re}\langle T(s)u(r) - u(s+r), u(s+r) - T(s)f \rangle. \end{aligned}$$

Since u is bounded, $M = \sup\{\|u(t)\| : t \geq 0\} < \infty$ and, since $\{T(s)f : s \geq 0\}$ is an almost-orbit of $\mathcal{S}$, $L = \sup\{\|T(s)f\| : s \geq 0\} < \infty$ by Lemma 2.2. Then it follows for all $t \geq s \geq t_0$ that

$$\begin{aligned} -\varepsilon &\leq \frac{1}{t} \int_0^t \|u(r) - T(s)f\|^2 dr + \|T(s)f - f\|^2 \\ &\quad + 2 \cdot \operatorname{Re}\langle S_t(u) - T(s)f, T(s)f - f \rangle - \frac{1}{t} \int_0^t \|u(s+r) - T(s)f\|^2 dr \\ &\quad - 2 \cdot \operatorname{Re} \frac{1}{t} \int_0^t \langle T(s)u(r) - u(s+r), u(s+r) - T(s)f \rangle dr \\ &\leq \frac{1}{t} \int_0^t \|u(r) - T(s)f\|^2 dr - \frac{1}{t} \int_s^{s+t} \|u(r) - T(s)f\|^2 dr + \|T(s)f - f\|^2 \\ &\quad + 2 \cdot \operatorname{Re}\langle S_t(u) - T(s)f, T(s)f - f \rangle \\ &\quad + \frac{2}{t} \int_0^t \|T(s)u(r) - u(s+r)\| \|u(s+r) - T(s)f\| dr \\ &\leq \frac{1}{t} \int_0^s \|u(r) - T(s)f\|^2 dr - \frac{1}{t} \int_t^{t+s} \|u(r) - T(s)f\|^2 dr + \|T(s)f - f\|^2 \\ &\quad + 2 \cdot \operatorname{Re}\langle S_t(u) - T(s)f, T(s)f - f \rangle + 2(M + \|T(s)f\|) \frac{1}{t} \int_0^t \varphi(r) dr \\ &\leq \frac{1}{t} \int_0^s \|u(r) - T(s)f\|^2 dr + \|T(s)f - f\|^2 \\ &\quad + 2 \cdot \operatorname{Re}\langle S_t(u) - T(s)f, T(s)f - f \rangle + 2(M + L) \frac{1}{t} \int_0^t \varphi(r) dr, \end{aligned}$$

that is,

$$\begin{aligned} -2 \cdot \operatorname{Re}\langle S_t(u) - T(s)f, T(s)f - f \rangle &\leq \frac{1}{t} \int_0^s \|u(r) - T(s)f\|^2 dr + \|T(s)f - f\|^2 \\ &\quad + 2(M + L) \frac{1}{t} \int_0^t \varphi(r) dr + \varepsilon. \end{aligned}$$

Now choose a sequence $\{t_n\}$, $t_n \uparrow \infty$, such that $S_{t_n}(u) \rightarrow f$ weakly. Substituting t_n for t in the last inequality above and taking the limit as $n \rightarrow \infty$, we arrive at $\|T(s)f - f\|^2 \leq \varepsilon(1 + 2M + 2L)$ for all $s \geq t_0$. This shows that $T(s)f \rightarrow f$ in norm and hence $T(s)f = f$ for all $t \geq 0$ by the continuity of the semigroup $\mathcal{S}$. This completes the proof. $\square$

By the same proof as above, we can prove

Corollary 3.1. *Let C and $\mathcal{F}$ be as in Theorem 3.1. Suppose u is a bounded almost-orbit of $\mathcal{F}$. Then for every real sequence $\{t_n\}$, $t_n \uparrow \infty$, and any $\{h_n\} \subset [0, \infty)$, every weak limit point of the sequence*

$$\frac{1}{t_n} \int_0^{t_n} u(r + h_n) dr, \quad n = 1, 2, \dots,$$

is a (common) fixed point of $\mathcal{F}$, i.e., an element in $F(\mathcal{F})$.

We now prove the main result in this paper.

Theorem 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H and $\mathcal{F} = \{T(t): t \geq 0\}$ be a continuous asymptotically nonexpansive semigroup acting on C such that each $T(t)$ is continuous and $F(\mathcal{F})$ is nonempty. Then for every almost-orbit u of $\mathcal{F}$, $\{Pu(t)\}$ converges to a fixed point z of $\mathcal{F}$ and $\{u(t)\}$ almost converges weakly to z , i.e.,*

$$\text{weak-} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t u(r + h) dr = z \quad \text{uniformly for } h \geq 0.$$

Proof. Since $F(\mathcal{F})$ is nonempty, $\{u(t)\}$ is bounded by Theorem 3.1 and Lemma 2.2 and $\{Pu(t)\}$ converges strongly to some z in $F(\mathcal{F})$ by Lemma 2.3. It remains only to prove that if $\{t_n\} \uparrow \infty$, $\{h_n\} \subset [0, \infty)$, and

$$\frac{1}{t_n} \int_0^{t_n} u(r + h_n) dr \rightarrow y$$

weakly, then $y = z$. To show this, let $M = \sup_{t \geq 0} \|Pu(t) - u(t)\|$. Since for every $r \geq 0$,

$$\operatorname{Re}\langle u(r) - Pu(r), Pu(r) - f \rangle \geq 0 \quad \text{for all } f \in F(\mathcal{F}),$$

we have

$$\begin{aligned} \operatorname{Re}\langle f - z, u(r) - Pu(r) \rangle &\leq \operatorname{Re}\langle Pu(r) - z, u(r) - Pu(r) \rangle \\ &\leq M \|Pu(r) - z\| \quad \text{for all } f \in F(\mathcal{F}). \end{aligned}$$

It follows that

$$\operatorname{Re} \left\langle f - z, \frac{1}{t_n} \int_0^{t_n} (u(r + h_n) - Pu(r + h_n)) dr \right\rangle \leq \frac{M}{t_n} \int_0^{t_n} \|Pu(r + h_n) - z\| dr.$$

Since $\lim_{t \rightarrow \infty} \|Pu(t) - z\| = 0$, taking the limit as n goes to infinity in the above inequality yields

$$\operatorname{Re}\langle f - z, y - z \rangle \leq 0 \quad \text{for all } f \in F(\mathcal{F});$$

since $y \in F(\mathcal{F})$, we then have $\|y - z\|^2 = \operatorname{Re}\langle y - z, y - z \rangle \leq 0$ and thus $y = z$. This completes the proof. $\square$

Theorem 3.3. *Under the same assumptions of Theorem 3.2, for each almost-orbit $\{u(t)\}$ of $\mathcal{F}$, the following statements are equivalent:*

- (i) $\{u(t)\}$ converges weakly to a fixed point of $\mathcal{F}$;
- (ii) $\{u(t)\}$ is weakly asymptotically regular; that is, $\{u(t + h) - u(t)\}$ converges weakly to zero as t tends to infinity for each $h \geq 0$;
- (iii) $\{u(t + h) - u(t)\}$ converges weakly to zero as t tends to infinity uniformly for $h \geq 0$.

Proof. By Lemma 2.2 and Theorem 3.1, we need only to prove (iii) $\Rightarrow$ (i). By Theorem 3.2, $\{u(t)\}$ almost converges weakly to $z = \lim_{t \rightarrow \infty} Pu(t)$. So for each $w \in H$ and $\varepsilon > 0$, we can choose a t_1 for which

$$\left| \left\langle w, \frac{1}{t_1} \int_0^{t_1} u(r+h) dr - z \right\rangle \right| < \frac{1}{2}\varepsilon \quad \text{for all } h \geq 0.$$

Then we choose a t_2 such that

$$|\langle w, u(t+r) - u(t) \rangle| < \frac{1}{2}\varepsilon \quad \text{for all } t > t_2 \text{ and } 0 \leq r \leq t_1.$$

Hence for $t > \max\{t_1, t_2\}$, we have

$$\begin{aligned} |\langle w, u(t) - z \rangle| &= \left| \left\langle w, \frac{1}{t_1} \int_0^{t_1} u(t) dr - z \right\rangle \right| \\ &\leq \left| \left\langle w, \frac{1}{t_1} \int_0^{t_1} u(r+t) dr - z \right\rangle \right| \\ &\quad + \left| \left\langle w, \frac{1}{t_1} \int_0^{t_1} (u(r+t) - u(t)) dr \right\rangle \right| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \end{aligned}$$

and the result follows. $\square$

4. NONEXPANSIVE RETRACTIONS

In this section we prove the existence of a nonexpansive retraction for almost-orbits of an asymptotically nonexpansive semigroup. We begin with

Lemma 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H and $\mathcal{F} = \{T(t): t \geq 0\}$ be an asymptotically nonexpansive semigroup acting on C . Suppose u is an almost-orbit of $\mathcal{F}$. Then for each $h \geq 0$, the function $v: [0, \infty) \rightarrow C$ defined by $v(t) = T(h)u(t)$ is also an almost-orbit of $\mathcal{F}$.*

Proof. As before, we set $\varphi(t) = \sup_{s \geq 0} \|u(s+t) - T(s)u(t)\|$. Since

$$\begin{aligned} \|v(s+t) - T(s)v(t)\| &= \|T(h)u(s+t) - T(s)T(h)u(t)\| \\ &\leq \|T(h)u(s+t) - u(h+s+t)\| + \|u(h+s+t) - T(s+h)u(t)\| \\ &\leq \varphi(s+t) + \varphi(t), \end{aligned}$$

the result follows. $\square$

Lemma 4.2. *Under the same assumptions of Lemma 4.1, the set*

$$\bigcap_{s \geq 0} \overline{\text{co}}\{u(t): t \geq s\} \cap F(\mathcal{F})$$

is either empty or the singleton set $\{z\}$ where $z = \text{weak-}\lim_{t \rightarrow \infty} (1/t) \int_0^t u(r) dr = \lim_{t \rightarrow \infty} Pu(t)$.

Proof. If w is in $\bigcap_{s \geq 0} \overline{\text{co}}\{u(t): t \geq s\} \cap F(\mathcal{F})$ then $F(\mathcal{F})$ is nonempty, and hence by Theorem 3.2 and Lemma 2.4, $z := \text{weak-}\lim_{t \rightarrow \infty} (1/t) \int_0^t u(r) dr = \lim_{t \rightarrow \infty} Pu(t)$ is the asymptotic center of $\{u(t)\}$ in $F(\mathcal{F})$. We conclude the

proof by showing $w = z$. By Lemma 2.2, $r(w) := \lim_{t \rightarrow \infty} \|u(t) - w\|$ and $r := \lim_{t \rightarrow \infty} \|u(t) - z\|$ exist and $r(w) \geq r$. Now, since

$$\|z - w\|^2 = \|u(s) - w\|^2 - \|u(s) - z\|^2 - 2 \cdot \operatorname{Re}\langle z - w, u(s) - z \rangle,$$

we get

$$\begin{aligned} \|z - w\|^2 + 2 \cdot \lim_{s \rightarrow \infty} \operatorname{Re}\langle z - w, u(s) - z \rangle \\ = \lim_{s \rightarrow \infty} \|u(s) - w\|^2 - \lim_{s \rightarrow \infty} \|u(s) - z\|^2 = r(w)^2 - r^2 \geq 0. \end{aligned}$$

For any $\varepsilon > 0$, there is a s_0 so large that for $s \geq s_0$,

$$2 \cdot \operatorname{Re}\langle z - w, u(s) - z \rangle > -\|z - w\|^2 - \varepsilon.$$

Since $w \in \overline{\operatorname{co}}\{u(t) : t \geq s_0\}$, it follows that

$$2 \cdot \operatorname{Re}\langle z - w, w - z \rangle \geq -\|z - w\|^2 - \varepsilon,$$

that is, $\|z - w\|^2 \leq \varepsilon$; since $\varepsilon > 0$ is arbitrary, we must have $z = w$. $\square$

Theorem 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H and let $\mathcal{F} = \{T(t) : t \geq 0\}$ be a continuous asymptotically nonexpansive semi-group acting on C such that each $T(t)$ is continuous and $F(\mathcal{F})$ is nonempty. Then the limit*

$$Q(u) := \operatorname{weak-}\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t u(r) dr$$

is the unique retraction from the set $\operatorname{AO}(\mathcal{F})$ of all almost-orbits of $\mathcal{F}$ onto $F(\mathcal{F})$ such that

(i) Q is nonexpansive in the sense that

$$\|Qu - Qv\| \leq \|u - v\|_\infty := \sup_{t \geq 0} \|u(t) - v(t)\| \quad \text{for } u, v \in \operatorname{AO}(\mathcal{F});$$

(ii) $QT(h)u = T(h)Qu = Qu$ for $u \in \operatorname{AO}(\mathcal{F})$ and $h \geq 0$; and

(iii) $Q(u) \in \bigcap_{s \geq 0} \overline{\operatorname{co}}\{u(t) : t \geq s\}$ for $u \in \operatorname{AO}(\mathcal{F})$.

Proof. (i) By the weak lower semicontinuity of the norm of H , we derive that

$$\|Qu - Qv\| \leq \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t \|u(r) - v(r)\| dr \leq \|u - v\|_\infty;$$

that is, Q is nonexpansive and (i) is proven.

(ii) Let $u \in \operatorname{AO}(\mathcal{F})$ and $h \geq 0$. First we observe that $QT(h)u$ is well defined by Lemma 4.1. Since $Qu \in F(\mathcal{F})$, we have $T(h)Qu = Qu$. Thus it remains to prove $QT(h)u = Qu$. In fact, we have

$$\begin{aligned} QT(h)u &= \operatorname{weak-}\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t T(h)u(r) dr \\ &= \operatorname{weak-}\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t u(h+r) dr \\ &\quad + \operatorname{weak-}\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (T(h)u(r) - u(h+r)) dr \\ &= \operatorname{weak-}\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t u(r+h) dr = Qu \end{aligned}$$

since

$$\frac{1}{t} \int_0^t \|T(h)u(r) - u(h+r)\| dr \leq \frac{1}{t} \int_0^t \varphi(r) dr \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

where

$$\varphi(t) = \sup_{s \geq 0} \|u(s+t) - T(s)u(t)\| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Finally, (iii) follows directly from Theorem 3.2. The proof is complete. $\square$

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