CHERN-SIMONS-MASLOV CLASSES OF SOME SYMPLECTIC VECTOR BUNDLES

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ABSTRACT. Let E_0 , J_0 , and L_0 be the symplectic $2n$ -vector bundle, the compatible complex operator, and the Lagrangian subbundle that are determined by the $U(n)$ -extension of the principal $O(n)$ -bundle $U(n) \rightarrow U(n)/O(n)$. We compute the Chern-Simons-Maslov class $\mu^1(E_0, J_0, L_0)$. Then for a trivial symplectic $2n$ -bundle E , a compatible complex operator J , and a Lagrangian subbundle L , we compute Chern-Simons-Maslov classes $\mu^h(E, J, L)$ under some condition on the base space of E .

INTRODUCTION

A symplectic vector bundle E over a smooth manifold M has a compatible complex structure J . If E has a Lagrangian subbundle L then the Chern-Simons-Maslov classes $\mu^h(E, J, L)$ are defined by Vaisman [V].

If E is a complex extension of a real n -vector bundle L then it is canonically a symplectic $2n$ -vector bundle with a Lagrangian subbundle L . Suppose E is trivial. Then it is classified by the principal $U(n)$ -bundle

$$\pi_0 : P_0 = U(n) \times_{O(n)} U(n) \rightarrow M_0 = U(n)/O(n).$$

Let E_0 be the symplectic $2n$ -vector bundle corresponding to the $U(n)$ -bundle $P_0 \rightarrow M_0$, J_0 the natural complex structure in E_0 , and L_0 the Lagrangian subbundle of E_0 corresponding to the $O(n)$ -bundle $\pi_0 : U(n) \rightarrow M_0$. In this situation, we denote P_0 by $U_{J_0}(E_0)$. In this note, we compute explicitly the class $\mu^1(E_0, J_0, L_0)$ and the classes $\mu^h(E, J, L)$ for symplectic vector bundles E classified by E_0 , under some conditions on the base space M .

Since $U_{J_0}(E_0)$ is a trivial $U(n)$ -bundle by the off-diagonal section $s_0 : M_0 \rightarrow U_{J_0}(E_0)$, the cohomology ring $H^*(U_{J_0}(E_0); \mathbf{R})$ is isomorphic to the tensor product $H^*(M_0; \mathbf{R}) \otimes H^*(U(n); \mathbf{R})$ by the Künneth formula.

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Theorem 1. *Let $u \in H^1(M_0) \cong \mathbf{Z}$ and $v \in H^1(U(n)) \cong \mathbf{Z}$ be canonical generators. Then we have*

$$\mu^1(E_0, J_0, L_0) = \frac{1}{2}(u \otimes 1 - 2 \otimes v).$$

Let $y_k \in H^{2k-1}(U(n); \mathbf{R})$ be canonical generators of the cohomology ring $H^*(U(n); \mathbf{R})$ and $\Delta_* y_{2h-1} \in H^{4h-3}(U(n)/O(n); \mathbf{R})$ denote the canonical generators of the cohomology ring $H^*(U(n)/O(n); \mathbf{R})$. If E is trivial and it is classified by a map $f: M \rightarrow U(n)/O(n)$, then the pullbacks $\Delta_{E*} y_{2h-1} = f^* \Delta_* y_{2h-1}$ are Maslov classes in the sense of Kamber and Tondeur [KT].

Let $U_J(E)$ denote the total space of the principal $U(n)$ -bundle associated with E . It follows that $H^*(U_J(E); \mathbf{R})$ is isomorphic to $H^*(M; \mathbf{R}) \otimes H^*(U(n); \mathbf{R})$ as $H^*(U_{J_0}(E_0); \mathbf{R})$.

Theorem 2. *Let E be a symplectic $2n$ -vector bundle over a manifold M , being trivial with respect to a compatible complex structure J , and $L \subset E$ a Lagrangian subbundle. If $H^i(M; \mathbf{R}) = 0$ for $0 < i < 4h - 3$ then we have*

$$\mu^h(E, J, L) = -\Delta_{E*} y_{2h-1} \otimes 1 + 1 \otimes y_{2h-1}.$$

In §1, by following Vaisman [V], we review symplectic $2n$ -vector bundles E , compatible complex structures J , Lagrangian subbundles L , and Chern-Simons-Maslov classes $\mu^h(E, J, L)$. In §2, by making use of a flat connection on $U_{J_0}(E_0)$ extending the canonical $O(n)$ -connection of the bundle $U(n) \rightarrow M_0$, representative cocycles of $\mu^h(E_0, J_0, L_0)$ are obtained. In §3, Theorem 1 is proved. In the last section we show relations between the Chern-Simons-Maslov classes and the Maslov classes in the sense of Kamber and Tondeur and prove Theorem 2.

1. SYMPLECTIC VECTOR BUNDLES

Let $\pi_E: E \rightarrow M$ be a C^∞ symplectic $2n$ -vector bundle over a manifold M of dimension m with its structure defined by a nondegenerate cross-section Ω of $\bigwedge^2 E^*$, where E^* is the dual bundle of E . By taking a complex structure J compatible with Ω , E becomes a complex n -vector bundle. The principal $\mathrm{GL}(n, \mathbf{C})$ -bundle associated with E has a $U(n)$ -reduction, that is, a principal $U(n)$ -bundle $\pi: U_J(E) \rightarrow M$.

Let $L \subset E$ be a Lagrangian subbundle, i.e., a real n -dimensional subbundle with $\Omega|_L = 0$. Then the structural group of E is reduced to $O(n)$ and one gets a principal $O(n)$ -subbundle of $U_J(E)$, $\pi: U_J(E, L) \rightarrow M$ with equivalent unitary frames $\tilde{e}_i = (e_i - \sqrt{-1}J e_i)/\sqrt{2}$ for orthonormal basis $e_i \in L$, $i = 1, \dots, n$.

An $O(n)$ -connection θ on $U_J(E, L)$ is extended to a $U(n)$ -connection on $U_J(E)$ naturally. Since the Chern polynomials of odd order vanish on the Lie algebra of $O(n)$, for the curvature form $\Theta = d\theta + [\theta, \theta]$ we have $c_{2h-1}(\Theta) = 0$. $c_k(\Theta)$ is denoted by $\Delta(\theta)c_k$ when the connection θ is emphasized. It is also denoted by $c_k(\Theta^{(k)})$ showing the k -multilinear property.

Let Δ^1 denote the standard 1-simplex. $t\theta$ for $t \in \Delta^1$ is a differential 1-form on $U_J(E) \times \Delta^1$, which is not necessarily a connection on the $U(n)$ -bundle $U_J(E) \times \Delta^1 \rightarrow M \times \Delta^1$. We take a "formal curvature" of $t\theta$,

$$\overline{\Theta} = d(t\theta) + [t\theta, t\theta] = dt \wedge \theta + \overline{\Theta}_t$$

where $\bar{\Theta}_t$ is defined by $\bar{\Theta}_t = t d\theta + t^2[\theta, \theta] = t\Theta + t(t-1)[\theta, \theta]$.

A suspension form of the Chern form $\Delta(\theta)c_{2h-1} = c_{2h-1}(\Theta)$ is defined by

$$\begin{aligned} T(\theta)c_{2h-1} &= - \int_{\Delta^1} c_{2h-1}(\bar{\Theta}^{(2h-1)}) \\ &= (2h-1) \int_0^1 c_{2h-1}(\theta, \bar{\Theta}_t^{(2h-2)}) dt \end{aligned}$$

where $\int_{\Delta^1}$ means an integration over the fiber of the Δ^1 -bundle $M \times \Delta^1 \rightarrow M$.

By Chern and Simons [CS], one obtains

$$d(T(\theta)c_{2h-1}) = \Delta(\theta)c_{2h-1} = 0,$$

and the de Rham cohomology classes $[T(\theta)c_{2h-1}] \in H_{\text{DR}}^{4h-3}(U_J(E)) \cong H^{4h-3}(U_J(E); \mathbb{R})$ does not depend on the choice of an $O(n)$ -connection θ . This fact is due to the Δ^1 -bundle Stokes formula. These cohomology classes are called the *Chern-Simons-Maslov classes* of (E, J, L) and are denoted by $\mu^h(E, J, L)$.

2. THE $U(n)$ -BUNDLE $U(n) \times_{O(n)} U(n)$ ON M_0

The principal $O(n)$ -bundle $U(n) \rightarrow U(n)/O(n) = M_0$ determines an n -dimensional Euclidean vector bundle $L_0 \rightarrow M_0$. This is canonically extended to an n -dimensional Hermitian vector bundle $\pi_{E_0}: E_0 \rightarrow M_0$ with a complex operator J_0 .

E_0 is a symplectic $2n$ -vector bundle with the skew 2-form $\Omega_0(v, w) = -\text{Im } h_0(v, w)$ where h_0 is the Hermitian form of E_0 , $\text{Im } h_0$ is its imaginary part, and v, w are vectors of E_0 . L_0 is identified with a Lagrangian subbundle of E_0 .

The principal $U(n)$ -bundle associated with E_0 is

$$\pi_0: U_{J_0}(E_0) = U(n) \times_{O(n)} U(n) \rightarrow M_0.$$

By the section $s_0: M_0 \rightarrow U(n) \times_{O(n)} U(n)$ induced from the off-diagonal map $U(n) \rightarrow U(n) \times U(n)$, $g \mapsto (g, g^{-1})$, it follows that the principal $U(n)$ -bundle $U_{J_0}(E_0)$ is trivial; that is, we have a bundle isomorphism

$$U(n) \times_{O(n)} U(n) \cong M_0 \times U(n).$$

Under this isomorphism, the $O(n)$ -subbundle associated with L_0 goes over to $\bar{\delta}(U(n)) \subset M_0 \times U(n)$ according to the reduction argument of a structural group, where the map $\bar{\delta}: U(n) \rightarrow M_0 \times U(n)$ is defined by $\bar{\delta}g = (\pi_0(g), g)$. Indeed, $\bar{\delta}$ is a bundle map covering the diagonal map δ of M_0 , and we obtain a commutative diagram:

$$\begin{array}{ccc} U(n) & \xrightarrow{\bar{\delta}} & M_0 \times U(n) \\ \downarrow & & \downarrow \\ M_0 & \xrightarrow{\delta} & M_0 \times M_0 \end{array}$$

Let $\mathfrak{u}(n)$ be the Lie algebra of $U(n)$ and $\mathfrak{o}(n)$ that of $O(n)$. Let ω be the $\mathfrak{u}(n)$ -valued Maurer-Cartan form on $U(n)$ and ω_0 its $\mathfrak{o}(n)$ -component. ω_0 defines an $O(n)$ -connection on the $O(n)$ -bundle $U(n) \rightarrow M_0$. Let θ be the natural extension of ω_0 on the $U(n)$ -bundle $U_{J_0}(E_0)$.

Lemma 3. *A $\mathfrak{u}(n)$ -valued 1-form $\omega - \omega_0$ on $U_{J_0}(E_0)$ is the $O(n)$ -basic part of ω .*

Proof. Since θ is the extension of ω_0 , it follows that for $X \in T(U(n))$

$$\begin{aligned}\omega_0(X) &= \bar{\delta}^* \theta(X) = \theta(\bar{\delta}_*(X)) = \theta((\pi_{0*}(X), X)) \\ &= \theta((\pi_{0*}(X), 0)) + \theta((0, X)) = \theta(s_{0*} \pi_{0*}(X)) + \omega(X).\end{aligned}$$

Therefore, we get

$$(\omega - \omega_0)(X) = \omega(X) - \omega_0(X) = -\pi_0^* s_0^* \theta(X),$$

which means that $\omega - \omega_0$ is annihilated by $\mathfrak{o}(n)$ and invariant under the right translation by $O(n)$. $\square$

Since ω_0 is obviously flat, its $U(n)$ -extension θ is flat too, and hence the curvature Θ of θ is 0. Thus it follows that

$$\begin{aligned}T(\theta)c_{2h-1} &= \frac{(-1)^{h+1}\sqrt{-1}(2h-1)}{(2h-1)!(2\pi)^{2h-1}} \int_0^1 \left\{ \sum \delta_{i_1 \dots i_{2h-1}}^{j_1 \dots j_{2h-1}} \theta_{j_1}^{i_1} \wedge \bar{\Theta}_{j_2}^{i_2} \wedge \dots \wedge \bar{\Theta}_{j_{2h-1}}^{i_{2h-1}} \right\} \wedge dt \\ &= \frac{(-1)^{h+1}\sqrt{-1}(2h-1)}{(2h-1)!(2\pi)^{2h-1}} \\ &\quad \times \int_0^1 \left\{ \sum \delta_{i_1 \dots i_{2h-1}}^{j_1 \dots j_{2h-1}} \theta_{j_1}^{i_1} \wedge (t\Theta_{j_2}^{i_2} + t(t-1)[\theta, \theta]_{j_2}^{i_2}) \right. \\ &\quad \left. \wedge \dots \wedge (t\Theta_{j_{2h-1}}^{i_{2h-1}} + t(t-1)[\theta, \theta]_{j_{2h-1}}^{i_{2h-1}}) \right\} \wedge dt \\ &= \frac{(2h-1)!(2h-1)!}{(4h-3)!} \frac{(-1)^{h+1}\sqrt{-1}}{(2h-1)!(2\pi)^{2h-1}} \\ &\quad \times \left\{ \sum \delta_{i_1 \dots i_{2h-1}}^{j_1 \dots j_{2h-1}} \theta_{j_1}^{i_1} \wedge [\theta, \theta]_{j_2}^{i_2} \wedge \dots \wedge [\theta, \theta]_{j_{2h-1}}^{i_{2h-1}} \right\} \\ &= \Delta(\theta)\sigma c_{2h-1},\end{aligned}$$

where σc_k is of the form

$$\frac{(k-1)!k!}{(2k-1)!} c_k(\iota \wedge \underbrace{[\iota, \iota] \wedge \dots \wedge [\iota, \iota]}_{k-1})$$

for the identity 1-form ι on $\mathfrak{u}(n)$ and is denoted by y_k .

It is well known that $H^*(U(n); \mathbf{R}) \cong H^*(\mathfrak{u}(n)) \cong \bigwedge(y_1, \dots, y_n)$ and for $n' = 2[(n+1)/2] - 1$ that

$$H^*(U(n)/O(n); \mathbf{R}) \cong H^*(\mathfrak{u}(n), O(n)) \cong \bigwedge(y_1, y_3, \dots, y_{n'}).$$

Hereinafter, we denote y_{2h-1} in $H^*(U(n)/O(n); \mathbf{R})$ by $\Delta_* y_{2h-1}$.

3. y_k , $\Delta_* y_{2h-1}$, AND THE CLASS $\mu^1(E_0, J_0, L_0)$

In the trivialization, $U_{J_0}(E_0) \cong M_0 \times U(n)$, let $i_0: U(n) \rightarrow U_{J_0}(E_0)$ be the inclusion map defined by $g \mapsto ([I_n], g)$ for $g \in U(n)$, where I_n is the unit element of $U(n)$.

Lemma 4. Let θ be the natural extension of the $O(n)$ -connection ω_0 on $U(n) \rightarrow M_0$ to $U_{J_0}(E_0)$. We have $s_0^*[T(\theta)c_{2h-1}] = -\Delta_* y_{2h-1}$ and $i_0^*[T(\theta)c_k] = y_k$.

Proof. Since $\pi_0^* s_0^* \theta$ is the $O(n)$ -basic part of ω by Lemma 3, it follows that

$$s_0^*[T(\theta)c_{2h-1}] = [T(s_0^* \theta)c_{2h-1}] = s_0^*[T(\pi_0^* s_0^* \theta)c_{2h-1}] = -\Delta_* y_{2h-1}.$$

Since θ is the $U(n)$ -connection, we have

$$i_0^*[T(\theta)c_k] = [T(i_0^* \theta)c_k] = [T(\omega)c_k] = y_k. \quad \square$$

It is known that $\pi_1(U(n)) \cong \mathbf{Z}$ and $\pi_1(U(n)/O(n)) \cong \mathbf{Z}$ (see, e.g., [M]), and hence we get $H^1(U(n)) \cong \mathbf{Z}$ and $H^1(U(n)/O(n)) \cong \mathbf{Z}$.

Proof of Theorem 1. By the Künneth formula for cohomology rings (see, e.g., [S]), we have $H^1(U_{J_0}(E_0); \mathbf{R}) \cong H^1(M_0 \times U(n); \mathbf{R}) \cong H^1(M_0; \mathbf{R}) \otimes 1 + 1 \otimes H^1(U(n); \mathbf{R})$, and the identification formula for $\mu \in H^1(U_{J_0}(E_0); \mathbf{R})$ under this isomorphism is $\mu = s_0^* \mu \otimes 1 + 1 \otimes i_0^* \mu$. It follows that

$$s_0^* \mu^1(E_0, J_0, L_0) = s_0^*[T(\theta)c_1] = \left[\frac{\sqrt{-1}}{2\pi} \text{trace}(s_0^* \theta) \right].$$

The closed path $\gamma: [0, \pi] \rightarrow U(n)/O(n)$ defined by $\gamma(t) = [e^{\sqrt{-1}t} \oplus I_{n-1}]$ represents the canonical generator $a \in H_1(U(n)/O(n))$. Let $\tilde{\gamma}$ be a lift of γ to $U(n)$. Then we have

$$\begin{aligned} \left[\frac{\sqrt{-1}}{2\pi} \text{trace}(s_0^* \theta) \right] (a) &= \frac{\sqrt{-1}}{2\pi} \int_{\gamma} \text{trace}(s_0^* \theta) = -\frac{\sqrt{-1}}{2\pi} \int_{\tilde{\gamma}} \text{trace} \omega \\ &= -\frac{\sqrt{-1}}{2\pi} \int_0^\pi \sqrt{-1} dt = \frac{1}{2}, \end{aligned}$$

and hence $s_0^* \mu^1(E_0, J_0, L_0) = -\frac{1}{2}u$.

Similarly, it follows that

$$i_0^* \mu^1(E_0, J_0, L_0) = i_0^*[T(\theta)c_1] = [T(\omega)c_1] = -v.$$

Thus one obtains

$$\begin{aligned} \mu^1(E_0, J_0, L_0) &= s_0^* \mu^1(E_0, J_0, L_0) \otimes 1 + 1 \otimes i_0^* \mu^1(E_0, J_0, L_0) \\ &= \frac{1}{2}(u \otimes 1 - 2 \otimes v). \quad \square \end{aligned}$$

4. THE CLASSES $\mu^h(E, J, L)$

Let $\pi: E \rightarrow M$ be a symplectic $2n$ -vector bundle, J a compatible complex structure, and $L \subset E$ a Lagrangian subbundle. Suppose that E is trivial as the complex n -vector bundle with respect to J . Then it is classified by a map $f: M \rightarrow U(n)/O(n)$, and we have the commutative diagram

$$\begin{array}{ccc} U_J(E) = E \times_{O(n)} U(n) & \xrightarrow{\tilde{f}} & U_{J_0}(E_0) = U(n) \times_{O(n)} U(n) \\ \pi \updownarrow s & & \pi_0 \updownarrow s_0 \\ M & \xrightarrow{f} & M_0 = U(n)/O(n) \end{array}$$

where $\tilde{f}$ is the bundle space map covering f and s is the section pulled back from s_0 by f .

High-dimensional Maslov classes for (E, J, L) in the sense of Kamber and Tondeur [KT] are the pullbacks $\Delta_* y_{2h-1} = f^* \Delta_* y_{2h-1} \in H^{4h-3}(M; \mathbf{R})$.

Proposition 5. We have $s^*\mu^h(E, J, L) = -\Delta_{E*}y_{2h-1}$.

Proof. The pullback $\tilde{f}^*\theta$ is a flat connection on $U_J(E)$. From the commutativity of the above diagram and the first formula of Lemma 4, it follows that

$$\begin{aligned} s^*\mu^h(E, J, L) &= s^*[T(\tilde{f}^*\theta)c_{2h-1}] = s^*\tilde{f}^*[T(\theta)c_{2h-1}] = f^*s_0^*[T(\theta)c_{2h-1}] \\ &= -f^*\Delta_*y_{2h-1} = -\Delta_{E*}y_{2h-1}. \quad \square \end{aligned}$$

Proposition 5 shows that the Maslov classes $\Delta_{E*}y_{2h-1}$ of Kamber and Tondeur are pullbacks of $-\mu^h(E, J, L)$ by the section of trivialization.

Proof of Theorem 2. We take a classifying map $f: M \rightarrow U(n)/O(n)$ for the $U(n)$ -bundle $\pi: U_J(E) \rightarrow M$ such that $f(x_b) = [I_n]$ for a fixed point $x_b \in M$. Let $i: U(n) \rightarrow U_J(E)$ be the inclusion map to the fiber over x_b . The section s gives a trivialization $U_J(E) \cong M \times U(n)$. One gets a commutative diagram:

$$\begin{array}{ccc} & U(n) & \\ i \swarrow & & \searrow i_0 \\ U_J(E) & \xrightarrow{\tilde{f}} & U_{J_0}(E_0) \end{array}$$

It follows that

$$\begin{aligned} i^*\mu^h(E, J, L) &= i^*[T(\tilde{f}^*\theta)c_{2h-1}] = [T(i^*\tilde{f}^*\theta)c_{2h-1}] = [T(i_0^*\theta)c_{2h-1}] \\ &= [T(\omega)c_{2h-1}] = y_{2h-1}, \end{aligned}$$

as the second formula of Lemma 4.

In the cohomology ring $H^*(U_J(E); \mathbf{R}) \cong H^*(M; \mathbf{R}) \otimes H^*(U(n); \mathbf{R})$, the $(4h-3, 0)$ -component of $\mu^h(E, J, L)$ is $s^*\mu^h(E, J, L) \otimes 1 = -\Delta_{E*}y_{2h-1} \otimes 1$ by Proposition 5 and the $(0, 4h-3)$ -component of $\mu^h(E, J, L)$ is $1 \otimes i^*\mu^h(E, J, L) = 1 \otimes y_{2h-1}$. The middle components of $\mu^h(E, J, L)$ all vanish by our assumption. Thus one gets

$$\mu^h(E, J, L) = -\Delta_{E*}y_{2h-1} \otimes 1 + 1 \otimes y_{2h-1}. \quad \square$$

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