DISCRETIZATION OF SECOND-ORDER VARIATIONAL SYSTEMS

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ABSTRACT. The existence of periodic orbits of the discretization of second-order variational systems is studied both for varying step of discretization and for a fixed small one.

1. INTRODUCTION

The purpose of this paper is to study the existence of periodic orbits of the mapping

$$(1.1) \quad \phi_h: (x, y) \rightarrow (2 \cdot x - y + h^2 \cdot f(x), x),$$

where $(x, y) \in R^m \times R^m$, $h > 0$, $f: R^m \rightarrow R^m$ is C^1 -smooth, $f = \text{grad } g$. We note that (1.1) presents the Euler-discretization of the equation

$$(1.2) \quad z'' = f(z)$$

with the step of magnitude h . Indeed, the discretization of (1.2) has the form

$$(1.3) \quad z_{n+1} - 2 \cdot z_n + z_{n-1} = h^2 \cdot f(z_n)$$

and putting $z_n = x_n$, $z_{n-1} = y_n$ we have

$$(1.4) \quad \begin{aligned} x_{n+1} &= 2 \cdot x_n - y_n + h^2 \cdot f(x_n), \\ y_{n+1} &= x_n. \end{aligned}$$

Hence periodic orbits of ϕ_h are periodic orbits of the Euler-discretization of (1.2) with the step h .

In §2 we investigate local behaviours of (1.1). We shall assume that $f(0) = 0$, $Df(0)$ has eigenvalues $-\alpha_1^2 \leq \dots \leq -\alpha_{m-s}^2 < 0 < \beta_1^2 \leq \dots \leq \beta_s^2$. By a result of Berger [3] we know that the system (1.2) has $m-s$ one-parameter families of periodic solutions with periods tending to $2\pi/\alpha_i$ and with amplitudes tending to 0. The next natural question arises: whether the mapping (1.1) has, for fixed h , $m-s$ families of invariant circles with amplitudes tending to 0 and rotation numbers tending to $2\pi/\alpha_i h, \dots, 2\pi/\alpha_{m-s} h$. By the remarks below the proof of Theorem 2.1 of this paper, we know that for h small the mapping ϕ_h has at 0 a center manifold of dimension $2(m-s)$ corresponding to $2(m-s)$ nonreal

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eigenvalues on the unit circle and stable and unstable manifolds of dimensions s . But easy computations show that the same property holds for (1.2). Indeed, the linearization of (1.2) at 0 has the eigenvalues $\pm l \cdot \alpha_1, \dots, \pm l \cdot \alpha_{m-s}$ and $\pm \beta_1, \dots, \pm \beta_s$. This correspondence may support a positive answer of the question. The author is not able to give this positive answer, because it seems that this problem is very difficult. Instead of this, we shall vary h and thus h will be a bifurcation parameter. We show (see Theorem 2.1) that there is an interval $(0, \omega)$ such that the set of bifurcation values of h of periodic orbits of (1.1) near 0 is dense in this interval. Of course, this result is consistent with a positive answer to the question, because the existence of the above-mentioned families of invariant circles of (1.1) immediately implies this result for ω small. Actually, our proof shows that we can take $\omega = 2/\alpha_{m-s}$, and hence our result is not restricted to h small. In the last part of this section we fix h and globally study the existence of periodic orbits of (1.1). We are also stimulated by a similar result of Hamiltonian systems [7, 2]. In these papers the authors studied (1.2) for $f = f(z, t)$ depending also on t and f is 2π -periodic in t . They assumed that 0 is a trivial solution of such a system and f is asymptotically linear at infinity. Under some conditions they showed the existence of a nontrivial periodic solution. Similarly, we shall assume that f has only a finite number of zeros and f is asymptotically linear at infinity. Then under some conditions we show the existence of an infinite number of periodic orbits of (1.1) for h small fixed, with minimal periods tending to infinity.

On the other hand, we note that the map ϕ_h can be considered as a higher dimensional version of the Hénon map [6] and, moreover, ϕ_h is volume preserving. There is a relation between area preserving twist maps [1] and ϕ_h as well. Indeed, following [1, p. 355] we put

$$h(x, z) = -|x - z|^2 - h^2 \cdot g(z),$$

and define a map F as $F(x_0, y_0) = (x_1, y_1)$ iff

$$y_0 = \frac{\partial}{\partial x} h(x_0, x_1), \quad y_1 = -\frac{\partial}{\partial x_1} h(x_0, x_1).$$

Then

$$\begin{aligned} y_0 &= x_1 - x_0, \\ y_1 &= x_1 - x_0 + h^2 \cdot f(x_1); \end{aligned}$$

i.e.,

$$\begin{aligned} x_1 &= y_0 + x_0, \\ y_1 &= y_0 + h^2 \cdot f(x_0 + y_0). \end{aligned}$$

Since $y_k = x_{k+1} - x_k$, we have

$$x_{k+1} - 2 \cdot x_k + x_{k-1} = h^2 \cdot f(x_k).$$

Thus the map F generates the map ϕ_h . We point out that our approach is similar to [1], since we shall also seek critical points of a functional defined similarly as in [1, p. 354]. Critical points of this functional are periodic orbits of F . For finding critical points we use a bifurcation theorem for variational problems [4] in a proof of Theorem 2.1 and Theorem 1.2 from [7] for proving Theorem 3.1 in this paper.

Finally, let us consider the mapping on R^2

$$\omega_h: \begin{cases} x_1 = x_0 + y_0, \\ y_1 = y_0 + h^2 \cdot f(x_0 + y_0), \end{cases}$$

where $f(0) = 0$, $f \in C^4$, $f'(0) < 0$, and f is 1-periodic. Note that

$$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \cdot \phi_h = \omega_h \cdot \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

We can consider ω_h as a map on $S^1 \times R$. Applying the twist theorem of Moser [6], we see that most of invariant curves of the form $S^1 \times \{r\}$ of ω_0 survive for h small. Thus, the existence of a family of invariant curves around $(0, 0)$ approximating solutions of $x'' = f(x)$ is impossible. We note that $(0, 0)$ is an elliptic fixed point of ω_h for h small. Thus if ω_h is sufficiently smooth and $(0, 0)$ is a nondegenerate elliptic fixed point, then the dynamics of ω_h are well known on some small open neighbourhood U_h of $(0, 0)$. Unfortunately U_h must shrink to the y -axis as $h \rightarrow 0$. The advantage of our result is that we can predict some periodic orbits of ω_h for h varying by using only the linearizations of ω_h at $(0, 0)$.

2. LOCAL EXISTENCE

We assume that $f(0) = 0$ and $Df(0)$ has eigenvalues $-\alpha_1^2 \leq \dots \leq -\alpha_{m-s}^2 < 0 < \beta_1^2 \leq \dots \leq \beta_s^2$. The following theorem shows us the existence of periodic orbits of (1.1) near $(0, 0)$.

Theorem 2.1. *For each $h \in \langle 0, 2/\alpha_{m-s} \rangle$ and $\delta > 0$ there is a sequence $\{h_n\}_1^\infty \subset \langle 0, 2/\alpha_{m-s} \rangle$ such that $h_n \rightarrow h$ and ϕ_{h_n} has a nontrivial periodic orbit $\{y_1, \dots, y_n\}$ satisfying $\sup_i |y_i| < \delta$. Hence the set*

$$\{h \in \langle 0, 2/\alpha_{m-s} \rangle \mid \phi_h \text{ possesses an orbit with the minimal period } n \geq n_0 \\ \text{and amplitude smaller than } \delta\}$$

is dense in $\langle 0, 2/\alpha_{m-s} \rangle$ for each $n_0 > 0$, $\delta > 0$ fixed. (By the trivial orbit we mean $y_i = 0$, $i = 1, \dots, n$.)

Proof. Let $(z_1, \dots, z_n)$ be a periodic orbit of (1.3). Then

$$\begin{aligned} z_2 - 2 \cdot z_1 + z_n &= h^2 \cdot f(z_1) \\ &\vdots \\ z_n - 2 \cdot z_{n-1} + z_{n-2} &= h^2 \cdot f(z_{n-1}) \\ z_1 - 2 \cdot z_n + z_{n-1} &= h^2 \cdot f(z_n). \end{aligned} \tag{2.1}$$

We put as in [5]

$$\begin{aligned} D: (R^m)^n &\rightarrow (R^m)^n, & F: (R^m)^n &\rightarrow (R^m)^n, \\ D(z_1, \dots, z_n) &= (z_2 - 2 \cdot z_1 + z_n, \dots, z_1 - 2 \cdot z_n + z_{n-1}), \\ F(z_1, \dots, z_n) &= (f(z_1), \dots, f(z_n)), \end{aligned}$$

and hence (2.1) has the form

$$Dz = h^2 \cdot F(z), \quad z = (z_1, \dots, z_n). \tag{2.2}$$

We note that D is symmetric and F has a gradient form. Hence (2.2) also has a variational form. By our assumption, $z = 0$ is a solution of (2.2). We compute the linearization of (2.2) at $z = 0$,

$$(2.3) \quad Dv = h^2 \cdot DF(0)v,$$

where $DF(0) = (Df(0), \dots, Df(0))$. Since $Df(0)$ is symmetric by using a change of variables, we can assume that $Df(0)$ is diagonal possessing the form

$$Df(0) = \begin{pmatrix} -\alpha_1^2 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & -\alpha_{m-s}^2 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \beta_1^2 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \beta_s^2 \end{pmatrix}.$$

We are interested in eigenvalues of (2.3). It is clear that for this purpose it is sufficient to find the solution λ of the equation

$$(2.4) \quad \begin{aligned} z_2 - 2 \cdot z_1 + z_n &= h^2 \cdot \gamma \cdot z_1 + \lambda \cdot z_1 \\ &\vdots \\ z_1 - 2 \cdot z_n + z_{n-1} &= h^2 \cdot \gamma \cdot z_n + \lambda \cdot z_n, \end{aligned}$$

where $\gamma \in \{-\alpha_1^2, \dots, \beta_s^2\}$, $z_i \in \mathbb{R}$. But we know from [5] that the left side matrix of (2.4) has the eigenvalues

$$\{-4 \cdot \sin^2(\pi/n) \cdot j \mid j = 0, \dots, n-1\}.$$

Hence $\lambda \in \{-4 \cdot \sin^2(\pi/n) \cdot j - h^2 \cdot \gamma \mid j = 0, \dots, n-1\}$. But this means that if we consider h as a parameter in (2.1) then each value of h such that $-4 \cdot \sin^2(\pi/n) \cdot j - h^2 \cdot \gamma = 0$ is a bifurcation point (see [4]). Hence $\mathcal{M}_n = \{2 \cdot \sin(\pi/n) \cdot j / \alpha_i \mid i = 1, \dots, s, j = 0, \dots, n-1\}$ are bifurcation points of (2.1). Finally, we note that $\bigcup \mathcal{M}_n$ is dense in $\langle 0, 2/\alpha_{m-s} \rangle$ and the proof is finished.

We compute the linearization of (1.1) at $x = y = 0$

$$(2.5) \quad (x_1, y_1) \rightarrow (2 \cdot x_1 - y_1 + h^2 \cdot Df(0)x_1, x_1),$$

and since $Df(0)$ is diagonal, (2.5) has a coordinate form

$$(2.6) \quad (x_1, y_1) \rightarrow (2 \cdot x_1 - y_1 + h^2 \cdot \gamma \cdot x_1, x_1),$$

where $\gamma \in \{-\alpha_1^2, \dots, \beta_s^2\}$. The eigenvalues of (2.6) are

$$\lambda_{1/2} = (2 + h^2 \cdot \gamma \pm h \cdot \sqrt{h^2 \cdot \gamma^2 + 4 \cdot \gamma})/2.$$

We see that for $\gamma > 0$, $|\lambda_{1/2}| \neq 1$ and for $\gamma < 0$ and for $h > 2 \cdot \sqrt{-1/\gamma}$, $|\lambda_{1/2}| \neq 1$, but for $\gamma < 0$, $h \in (0, 2 \cdot \sqrt{-1/\gamma})$, $|\lambda_{1/2}| = 1$. Thus for $h \in (0, 2/\alpha_{m-s})$ the linearization of (1.1) at $x = y = 0$ has an eigenvalue with the absolute value 1 and for $h > 2/\alpha_{m-s}$ it has not such an eigenvalue. Hence $x = y = 0$ is a hyperbolic fixed point of ϕ_h only for $h > 2/\alpha_{m-s}$, and this implies

Theorem 2.2. *There is $\delta > 0$ and a neighbourhood U of $(0, 0) \in R^{2m}$ for each $h_0 > 2/\alpha_{m-s}$ such that for each $h \in (h_0 - \delta, h_0 + \delta)$ the map ϕ_h has no nontrivial periodic orbit in U .*

Summing up we can state that if h varies in $(2/\alpha_{m-s}, \infty)$ then the dynamics of ϕ_h near $(0, 0) \in R^{2m}$ does not change, but for $h \in (0, 2/\alpha_{m-s})$ there are bifurcations of periodic orbits in arbitrary small neighbourhood of $(0, 0)$.

3. A GLOBAL RESULT

In the previous section we study ϕ_h for varying h . Now we fix $h > 0$ and study the existence of periodic orbits of ϕ_h in $R^m \times R^m$. We must solve (2.2) for the existence of n -periodic orbits. We assume for f

- (i) f has only zero points $x_1, x_2, \dots, x_k$,
- (ii) $Df(x_i)$, $i = 1, \dots, k$, have nonzero eigenvalues,
- (iii) $|f(x) - Ax| / |x| \rightarrow 0$ as $|x| \rightarrow 0$ for a symmetric matrix $A \in \mathcal{L}(R^m)$ with $\det A \neq 0$.

Since $Df(x_i)$ and A are symmetric, they have only real nonzero eigenvalue, and let $\{d_1^p, \dots, d_m^p\}$ and $\{a_1, \dots, a_m\}$ be eigenvalues of $Df(x_p)$ and A , respectively, for $p = 1, \dots, k$.

Theorem 3.1. *Let us assume*

$$\sum_{a_j < 0} \sqrt{-a_j} \neq \sum_{d_j^p < 0} \sqrt{-d_j^p}$$

for each $p = 1, \dots, k$. Then for $h > 0$ sufficiently small there is a sequence of natural numbers $\{n_i\}_0^\infty$ such that $n_i < n_{i+1}$, $n_i \rightarrow \infty$ as $i \rightarrow \infty$ and ϕ_h has a periodic orbit with the minimal period n_i for each i . (If $\{a_i, a_i < 0\} = \emptyset$ then we put $\sum_{a_i < 0} = 0$.)

Proof. We shall apply the following theorem.

Theorem [7]. *Let $a: R^m \rightarrow R$ be a C^2 -function satisfying*

$$|\nabla a(x) - A_\infty x| / |x| \rightarrow 0 \quad \text{as } |x| \rightarrow \infty$$

for a symmetric nonsingular matrix $A_\infty \in \mathcal{L}(R^m)$. Suppose that a has critical points $\tilde{x}_1, \dots, \tilde{x}_k$ and all of them are nondegenerate. If $M^+(a''(\tilde{x}_i)) \neq M^+(A_\infty)$ for each i then a has another critical point. Here a'' is the Hessian and $M^+(B)$ is the positive Morse index of the symmetric matrix B .

Let n be odd. We have for our case

$$a(z) = \frac{\langle Dz, z \rangle}{2} - h^2 \cdot \sum_{i=1}^n g(z_i),$$

$$\tilde{x}_i = (x_i, \dots, x_i), \quad A_\infty(z) = D - h^2 \cdot (Az_1, \dots, Az_n).$$

Note that $\text{grad } g = f$. We shall compute $M^+(a''(\tilde{x}_i))$. Since $a''(\tilde{x}_i) = D - h^2 \cdot DF(\tilde{x}_i)$ where $DF(\tilde{x}_i) = (Df(x_i), \dots, Df(x_i))$, $Df(x_i)$ can be considered in a diagonal form having the eigenvalues $d_1^i, \dots, d_m^i$. We can show, as in the above proof, that $a''(\tilde{x}_i)$ has eigenvalues

$$\left\{ -4 \cdot \sin^2 \frac{\pi}{n} \cdot j - h^2 \cdot d_s^i \mid j = 0, \dots, \frac{(n-1)}{2}, \quad s = 1, \dots, m \right\}.$$

Similarly, A_∞ has eigenvalues

$$\left\{ -4 \cdot \sin^2 \frac{\pi}{n} \cdot j - h^2 \cdot a_i \mid j = 0, \dots, \frac{(n-1)}{2}, i = 1, \dots, m \right\}.$$

First of all, we must ensure that $a''(\tilde{x}_i)$ and A_∞ are nonsingular, i.e.,

$$-4 \cdot \sin^2 \frac{\pi}{n} \cdot j - h^2 \cdot d_s^i \neq 0, \quad -4 \cdot \sin^2 \frac{\pi}{n} \cdot j - h^2 \cdot a_i \neq 0.$$

Hence it must hold for $0 < h$ small and $d_s^i < 0$, $a_i < 0$

$$(3.1) \quad \begin{aligned} \frac{j}{n} &\neq \arcsin h \cdot \sqrt{\frac{-d_s^i}{2}} \cdot \frac{1}{\pi}, & j = 0, \dots, \frac{(n-1)}{2}, \\ \frac{j}{n} &\neq \arcsin h \cdot \sqrt{-a_i} \cdot \frac{1}{\pi}, & j = 0, \dots, \frac{(n-1)}{2}. \end{aligned}$$

We put $\tilde{d}_s^i = \arcsin h \cdot \sqrt{-d_s^i}/2 \cdot \frac{1}{\pi}$, $\tilde{a}_i = \arcsin h \cdot \sqrt{-a_i}/2 \cdot \frac{1}{\pi}$ for $d_s^i < 0$, $a_i < 0$. Now it is clear that we can find a sequence of prime numbers $\{n_i\}_0^\infty$ such that the relation (3.1) holds with $n = n_i$ and $n_i \rightarrow \infty$ as $i \rightarrow \infty$. Then for $n = n_b$

$$\begin{aligned} M^+(a''(\tilde{x}_i)) &= 2 \cdot \# \left\{ -4 \cdot \sin^2 \frac{\pi}{n_b} \cdot j - h^2 \cdot d_s^i > 0 \mid \frac{(n_b-1)}{2} \geq j > 0, \right. \\ &\quad \left. s = 1, \dots, m \right\} \\ &\quad + \# \{ d_s^i < 0 \mid s = 1, \dots, m \}, \\ M^+(A_\infty) &= 2 \cdot \# \left\{ -4 \cdot \sin^2 \frac{\pi}{n_b} \cdot j - h^2 \cdot a_s > 0 \mid \frac{(n_b-1)}{2} \geq j > 0, \right. \\ &\quad \left. s = 1, \dots, m \right\} \\ &\quad + \# \{ a_s < 0 \mid s = 1, \dots, m \}, \end{aligned}$$

since $-4 \cdot \sin^2(\pi/n_b) \cdot j$, $0 < j \leq (n_b-1)/2$, and 0 have the geometric multiplicity 2 and 1 respectively, in D for $m = 1$ (see [5]), and $\#$ denotes cardinality.

Hence

$$\begin{aligned} M^+(a''(\tilde{x}_i)) &= 2 \cdot \# \left\{ \frac{j}{n_b} < \tilde{d}_s^i \mid 0 < j \leq \frac{(n_b-1)}{2}, s = 1, \dots, m \right\} \\ &\quad + \# \{ d_s^i < 0 \mid s = 1, \dots, m \}, \\ M^+(A_\infty) &= 2 \cdot \# \left\{ \frac{j}{n_b} < \tilde{a}_s \mid 0 < j \leq \frac{(n_b-1)}{2}, s = 1, \dots, m \right\} \\ &\quad + \# \{ a_s < 0 \mid s = 1, \dots, m \}. \end{aligned}$$

For h small $\tilde{d}_s^i < \frac{1}{2}$, $\tilde{a}_i < 1$ and thus,

$$\begin{aligned} M^+(a''(\tilde{x}_i)) &= 2 \cdot \sum_{d_s^i < 0} \left([\tilde{d}_s^i \cdot n_b] + \frac{1}{2} \right), \\ M^+(A_\infty) &= 2 \cdot \sum_{a_i < 0} \left([\tilde{a}_i \cdot n_b] + \frac{1}{2} \right) \end{aligned}$$

($[x]$ is the integer part of x). We compute

$$M^+(a''(\tilde{x}_i)) - M^+(A_\infty) = 2 \cdot n_b \cdot h \cdot \left(\sum_{d_s^i < 0} \frac{\arcsin h \cdot \sqrt{-d_s^i}/2 \cdot 1/\pi}{h} - \sum_{a_i < 0} \frac{\arcsin h \cdot \sqrt{-a_i}/2 \cdot 1/\pi}{h} \right) + C,$$

where $|C| \leq 4 \cdot m$. Now using $\lim_{h \rightarrow 0} (\arcsin h \cdot d)/h = d$ and our assumption concerning d_s^i and a_i , we see that for $h > 0$ small fixed and n_b large

$$M^+(a''(\tilde{x}_i)) \neq M^+(A_\infty), \quad i = 1, \dots, k.$$

But this means that a has a nontrivial critical point, i.e., ϕ_h possesses an orbit with the minimal period n_b . Note that n_b is a large prime number. This finishes the proof.

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