A VOLTERRA TYPE DERIVATIVE OF THE LEBESGUE INTEGRAL

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(Communicated by Andrew M. Bruckner)

ABSTRACT. Using functions of bounded variation, we define a Volterra type derivative of the linear functional associated with a Lebesgue integrable function and show that it is equal to this function almost everywhere.

If Ω is an open subset of $\mathbf{R}^m$ and $f \in L^1_{\text{loc}}(\Omega)$, let

$$F(\theta) = \int_{\Omega} f\theta$$

for each $\theta \in L^\infty_0(\Omega)$, where $L^\infty_0(\Omega)$ is the linear space of all $\theta \in L^\infty(\Omega)$ that vanish almost everywhere outside a compact subset of Ω . It was shown in [P1, Corollary] that the Volterra derivative $\delta F/\delta\theta$ with respect to continuous functions θ displays a rather special behavior: it is continuous on the subset of Ω where it exists (cf. also [HN, Theorem 2]). In this note we show that using functions θ of bounded variation, a more natural type of the Volterra derivative $\mathfrak{D}F$ of F can be defined. By controlling a Vitali type regularity of the functions with respect to which we differentiate, we achieve that $\mathfrak{D}F = f$ almost everywhere in Ω whenever $f \in L^\infty_{\text{loc}}(\Omega)$. This result is used in [KMP] to show that a certain conditionally convergent integral in $\mathbf{R}^m$ is invariant with respect to lipeomorphisms.

Throughout, we fix an integer $m \geq 1$ and an open set $\Omega \subset \mathbf{R}^m$. The norms in $L^1(\Omega)$ and $L^\infty(\Omega)$ are denoted by $|\cdot|_1$ and $|\cdot|_\infty$, respectively. Unless stated otherwise, all functions and functionals we consider are real valued.

For $x = (\xi_1, \dots, \xi_m)$ in $\mathbf{R}^m$ and $\varepsilon > 0$, let $|x| = \sqrt{\xi_1^2 + \dots + \xi_m^2}$ and $U(x, \varepsilon) = \{y \in \mathbf{R}^m : |x - y| < \varepsilon\}$. If $E \subset \mathbf{R}^m$ then E^- , $d(E)$, and $|E|$ denote the closure, diameter, and the outer Lebesgue measure of E , respectively. Given a nonnegative function θ defined on Ω and $\alpha \geq 0$, we set $S_\theta(\alpha) = \{x \in \Omega : \theta(x) > \alpha\}$; we write S_θ instead of $S_\theta(0)$ and let $d(\theta) = d(S_\theta)$. We say that a function $\theta \in L^1(\Omega)$ is *normalized* whenever $|\theta|_1 = |S_\theta|$.

Received by the editors June 7, 1991; presented at the 15th Summer Symposium in Real Analysis, Czechoslovakia, August 1991.

1991 *Mathematics Subject Classification*. Primary 28A15; Secondary 46G05.

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Following [G, Definition 1.1], we say that $\theta \in L^1(\Omega)$ is of *bounded variation* if its distributional gradient $D\theta$ is a vector-valued Borel measure in Ω whose variation $|D\theta|$ is finite; we set $\|\theta\| = |D\theta|(\Omega)$. The family of all nonnegative functions $\theta \in L^\infty(\Omega)$ that are of bounded variation is denoted by BV_+ . The *regularity* of $\theta \in BV_+$ is the number

$$r(\theta) = \begin{cases} |\theta|_1/[d(\theta)\|\theta\|] & \text{if } d(\theta)\|\theta\| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma relates $r(\theta)$ to the usual concept of regularity of sets defined in [S, Chapter 4, §2].

Lemma 1. *There is a constant $c > 0$, depending only on the dimension m , such that*

$$\int_{S_\theta(\alpha)} \theta \leq c |S_\theta(\alpha)|^{1/m} \|\theta\|$$

for each $\theta \in BV_+$ and $\alpha \geq 0$. In particular,

$$\frac{r(\theta)}{c} \leq \frac{|S_\theta|^{1/m}}{d(S_\theta)} \quad \text{and} \quad \int_{S_\theta(\alpha)} \theta \leq \frac{c}{\sqrt[m]{\alpha}} \cdot \frac{|S_\theta|}{r(\theta)},$$

where the first inequality holds for each $\theta \in BV_+$ and the second for each normalized $\theta \in BV_+$ and $\alpha > 0$.

Proof. If $m \geq 2$ the Hölder and Sobolev inequalities yield

$$\int_{S_\theta(\alpha)} \theta \leq |S_\theta(\alpha)|^{1/m} \left(\int_{S_\theta} \theta^{m/(m-1)} \right)^{(m-1)/m} \leq c_m |S_\theta(\alpha)|^{1/m} \|\theta\|$$

where c_m is a positive constant depending only on m (see [G, Theorem 1.28]). If $m = 1$ then

$$\int_{S_\theta(\alpha)} \theta \leq |S_\theta(\alpha)| \cdot |\theta|_\infty \leq |S_\theta(\alpha)| \cdot \|\theta\|.$$

Thus letting $c_1 = 1$ establishes the first inequality. The second inequality follows immediately from the first with $\alpha = 0$. If θ is normalized then

$$\alpha |S_\theta(\alpha)| \leq \int_{S_\theta(\alpha)} \theta \leq |\theta|_1 = |S_\theta| \leq [d(\theta)]^m.$$

For $\alpha > 0$, the proof is completed by combining the first and last inequalities.

Definition. We say that a functional J on $L^\infty(\Omega)$ is *differentiable* at $x \in \Omega$ if there is a real number α such that given $\varepsilon > 0$ we can find a $\delta > 0$ so that

$$|J(\theta) - \alpha|\theta|_1| < \varepsilon|\theta|_1$$

for each $\theta \in BV_+$ with $x \in S_\theta^-$, $d(\theta) < \delta$, and $r(\theta) > \varepsilon$. If the inequality holds only when θ is, in addition, normalized and $|\theta|_\infty < 1/\varepsilon$, we say that J is *weakly differentiable* at x .

The number α from the definition is uniquely determined, and we denote it by $\mathfrak{D}J(x)$ or $\mathfrak{D}_w J(x)$ according to whether J is differentiable or weakly differentiable at x , respectively. If J is differentiable at x it is also weakly differentiable at x , and $\mathfrak{D}_w J(x) = \mathfrak{D}J(x)$. The example below shows, however, that the converse is false for $m > 1$. If $m = 1$, the two concepts of differentiability coincide since $|\theta|_\infty \leq \|\theta\| \leq 1/r(\theta)$ for each normalized $\theta \in BV_+$.

Proposition. Let $f \in L^1_{\text{loc}}(\Omega)$ and $F(\theta) = \int_{\Omega} f\theta$ for each $\theta \in L^\infty(\Omega)$. Suppose that F is weakly differentiable at $x \in \Omega$ and $\mathfrak{D}_w F(x) = f(x)$. If f is essentially bounded in a neighborhood of x , then F is differentiable at x and $\mathfrak{D}F(x) = f(x)$.

Proof. Choose $\delta > 0$ so that $\alpha = \sup_{y \in U(x, \delta)} \text{ess } |f(y)|$ is finite. Proceeding towards a contradiction, assume there is an $\varepsilon > 0$ and normalized functions $\theta_k \in BV_+$, $k = 1, 2, \dots$, such that $x \in S_{\theta_k}^-$, $d(\theta_k) < \delta/k$, $r(\theta_k) > \varepsilon$, and

$$\left| \int_{\Omega} f\theta_k - f(x) \int_{\Omega} \theta_k \right| > \varepsilon \int_{\Omega} \theta_k.$$

For $k, n = 1, 2, \dots$, let $\beta_{k,n} = |S_{\theta_k}| / |\min\{\theta_k, n\}|_1$ and $\theta_{k,n} = \beta_{k,n} \min\{\theta_k, n\}$. Each $\theta_{k,n}$ is a normalized function from BV_+ and $\|\theta_{k,n}\| \leq \beta_{k,n} \|\theta_k\|$. By Lemma 1, there is a constant $c > 0$, depending only on m , such that

$$|S_{\theta_k}| \geq \int_{\Omega} \min\{\theta_k, n\} = \int_{\Omega} \theta_k - \int_{S_{\theta_k}(n)} \theta_k \geq |S_{\theta_k}| \left(1 - \frac{c}{\varepsilon} \cdot \frac{1}{\sqrt[m]{n}} \right)$$

and

$$\begin{aligned} \left| \int_{\Omega} f\theta_k - \int_{\Omega} f\theta_{k,n} \right| &\leq \alpha \left[\int_{\Omega} |\theta_k - \beta_{k,n}\theta_k| + \int_{\Omega} |\beta_{k,n}\theta_k - \theta_{k,n}| \right] \\ &= \alpha \left[(1 - \beta_{k,n}) \int_{\Omega} \theta_k + \beta_{k,n} \int_{\Omega} (\theta_k - \min\{\theta_k, n\}) \right] \\ &= \alpha \left[(1 - \beta_{k,n}) |S_{\theta_k}| + \beta_{k,n} \int_{S_{\theta_k}(n)} \theta_k \right] \\ &\leq \alpha |S_{\theta_k}| \left(1 - \beta_{k,n} + \beta_{k,n} \frac{c}{\varepsilon} \cdot \frac{1}{\sqrt[m]{n}} \right). \end{aligned}$$

The first inequality yields

$$1 \geq \frac{1}{\beta_{k,n}} \geq 1 - \frac{c}{\varepsilon} \cdot \frac{1}{\sqrt[m]{n}}$$

and hence $\lim_{n \rightarrow \infty} \beta_{k,n} = 1$ uniformly in k . From this and the second inequality we see that

$$\lim_{n \rightarrow \infty} \frac{1}{|S_{\theta_k}|} \int_{\Omega} f\theta_{k,n} = \frac{1}{|S_{\theta_k}|} \int_{\Omega} f\theta_k$$

uniformly in k . Thus there is an integer $p \geq 1$ such that $\beta_{k,p} < 2$ and

$$\left| \frac{1}{|S_{\theta_k}|} \int_{\Omega} f\theta_{k,p} - f(x) \right| > \frac{\varepsilon}{2}$$

for $k = 1, 2, \dots$. Since $S_{\theta_k} = S_{\theta_{k,p}}$, we have $|S_{\theta_k}| = \int_{\Omega} \theta_{k,p}$ and

$$\left| \int_{\Omega} f\theta_{k,p} - f(x) \int_{\Omega} \theta_{k,p} \right| > \frac{\varepsilon}{2} \int_{\Omega} \theta_{k,p}.$$

As $\lim_{k \rightarrow \infty} d(\theta_{k,p}) = 0$, $r(\theta_{k,p}) > \varepsilon/2$, and $|\theta_{k,p}|_{\infty} \leq 2p$, a contradiction follows.

Lemma 2. Let $E \subset \mathbf{R}^m$ be a measurable set of finite measure, and let $g \in L^1(E)$. Given $\varepsilon > 0$, we can find a positive function δ on E so that

$$\sum_{i=1}^{\infty} \left| g(x_i) \int_E \nu_i - \int_E g \nu_i \right| < \varepsilon$$

for each collection $\{(\nu_i, x_i) : i = 1, 2, \dots\}$ such that $x_i \in E$, $\nu_i \geq 0$ is a measurable function on E , $S_{\nu_i} \subset U(x_i, \delta(x_i))$, $i = 1, 2, \dots$, and $\sum_{i=1}^{\infty} \nu_i \leq 1/\varepsilon$.

Proof. Choose an $\varepsilon > 0$ and let $\eta = \varepsilon^2/(1 + |E|)$. By the Vitali-Carathéodory theorem (see [R, Theorem 2.25]), there are extended real-valued functions h and k on E such that h is upper semicontinuous, k is lower semicontinuous, $h \leq g \leq k$, and $\int_E (k - h) < \eta$. Find a positive function δ on E so that $h(y) < g(x) + \eta$ and $k(y) > g(x) - \eta$ for each $x, y \in E$ with $|x - y| < \delta(x)$, and let $\{(\nu_i, x_i) : i = 1, 2, \dots\}$ be the collection from the lemma. Then

$$\int_E h \nu_i - \eta \int_E \nu_i \leq g(x_i) \int_E \nu_i \leq \int_E k \nu_i + \eta \int_E \nu_i,$$

and $\int_E h \nu_i \leq \int_E g \nu_i \leq \int_E k \nu_i$ for $i = 1, 2, \dots$. As $\sum_{i=1}^{\infty} \nu_i \leq 1/\varepsilon$, we obtain

$$\begin{aligned} \sum_{i=1}^{\infty} \left| g(x_i) \int_E \nu_i - \int_E g \nu_i \right| &\leq \sum_{i=1}^{\infty} \int_E (k - h) \nu_i + \eta \sum_{i=1}^{\infty} \int_E \nu_i \\ &\leq \frac{1}{\varepsilon} \int_E (k - h) + \frac{\eta}{\varepsilon} |E| < \varepsilon. \end{aligned}$$

Theorem. Let $f \in L^1_{\text{loc}}(\Omega)$ and $F(\theta) = \int_{\Omega} f \theta$ for each $\theta \in L^{\infty}_0(\Omega)$. For almost all $x \in \Omega$ the functional F is weakly differentiable at x and $\mathfrak{D}_w F(x) = f(x)$. In particular, F is differentiable almost everywhere in Ω whenever $f \in L^{\infty}_{\text{loc}}(\Omega)$.

Proof. With no loss of generality we may assume that $|\Omega| < +\infty$ and $f \in L^1(\Omega)$. Let N be the set of all $x \in \Omega$ at which either F is not weakly differentiable or $\mathfrak{D}_w F(x) \neq f(x)$. Given $x \in N$, there is a $\gamma(x) > 0$ such that for each $\delta > 0$ we can find a normalized $\theta \in BV_+$ with $x \in S_{\theta}^-$, $d(\theta) < \delta$, $r(\theta) > \gamma(x)$, $|\theta|_1 < 1/\gamma(x)$, and $|F(\theta) - f(x)|_1 \geq \gamma(x)|\theta|_1$. Fix an integer $n \geq 1$ and set $N_n = \{x \in N : \gamma(x) > 1/n\}$. Choose an $\varepsilon > 0$, and use Lemma 2 to find a positive function δ on Ω so that

$$\sum_{i=1}^{\infty} \left| f(x_i) \int_{\Omega} \nu_i - \int_{\Omega} f \nu_i \right| < \frac{\varepsilon}{2n}$$

for each collection $\{(\nu_i, x_i) : i = 1, 2, \dots\}$ such that $x_i \in \Omega$, ν_i is a non-negative measurable function on Ω , $S_{\nu_i} \subset U(x_i, \delta(x_i))$, $i = 1, 2, \dots$, and $\sum_{i=1}^{\infty} \nu_i \leq n$. Let $\mathcal{S}$ be the family of all closed sets $S \subset \Omega$ such that $d(S) < \delta(x_S)$ for an $x_S \in \Omega \cap N_n$, and there is a measurable function ν_S on Ω with $0 \leq \nu_S \leq n$, $S_{\nu_S} \subset S$, $\int_{\Omega} \nu_S \geq |S|/2$, and

$$\left| f(x_S) \int_{\Omega} \nu_S - \int_{\Omega} f \nu_S \right| \geq \frac{1}{n} \int_{\Omega} \nu_S.$$

Claim. The family $\mathcal{S}$ is a Vitali cover of N_n .

Assuming the claim is correct, there are disjoint sets $S_1, S_2, \dots$ in $\mathcal{S}$ such that $|N_n - \bigcup_{i=1}^{\infty} S_i| = 0$ (see [S, Chapter 4, Theorem (3.1)]). If $\nu_i = \nu_{S_i}$ and $x_i = x_{S_i}$, $i = 1, 2, \dots$, then $\sum_{i=1}^{\infty} \nu_i \leq n$ and hence

$$|N_n| \leq \sum_{i=1}^{\infty} |S_i| \leq 2 \sum_{i=1}^{\infty} \int_{\Omega} \nu_i \leq 2n \sum_{i=1}^{\infty} \left| f(x_i) \int_{\Omega} \nu_i - \int_{\Omega} f \nu_i \right| < \varepsilon.$$

It follows from the arbitrariness of ε that $|N_n| = 0$, and as $N = \bigcup_{n=1}^{\infty} N_n$ we see that $|N| = 0$.

Proof of the claim. Choose an $x \in N_n$ and a positive $\Delta \leq \delta(x)$. There is a normalized $\theta \in BV_+$ with $x \in S_{\theta}^-$, $d(\theta) < \Delta$, $r(\theta) > 1/n$, $|\theta|_{\infty} < n$, and

$$\left| \int_{\Omega} f\theta - f(x) \int_{\Omega} \theta \right| > \frac{1}{n} \int_{\Omega} \theta.$$

By Lemma 1, $|S_{\theta}|/[d(S_{\theta})]^m \geq 1/(nc)^m$ where $c > 0$ is a constant depending only on m . Moreover, there is a compact set $K \subset S_{\theta}$ such that $|K| \geq |S_{\theta}|/2$, $\int_K \theta \geq \frac{1}{2} \int_{\Omega} \theta = |S_{\theta}|/2$, and

$$\left| \int_K f\theta - f(x) \int_K \theta \right| \geq \frac{1}{n} \int_K \theta.$$

Since

$$\frac{|K \cup \{x\}|}{[d(K \cup \{x\})]^m} \geq \frac{|K|}{[d(S_{\theta})]^m} \geq \frac{1}{2(nc)^m},$$

it suffices to let $S = K \cup \{x\}$, $x_S = x$, and set $\nu_S(y) = \theta(y)$ if $y \in S$ and $\nu_S(y) = 0$ if $y \in \Omega - S$.

Example. Let $m = 2$, $\Omega = \mathbf{R}^2$, and let $\{x_1, x_2, \dots\}$ be an enumeration of a countable dense set $D \subset \Omega$. The characteristic function of $E \subset \Omega$ is denoted by χ_E . We define an $f \in L^1(\Omega)$ by setting

$$f(x) = \sum_{n=1}^{\infty} |x - x_n|^{-3/2} \chi_{U(x_n, 2^{-n})}(x)$$

for each $x \in \Omega - D$. Clearly, f is essentially bounded in no nonempty open subset of Ω , and we show that the functional $F : \theta \mapsto \int_{\Omega} f\theta$ on $L^{\infty}(\Omega)$ is nowhere differentiable. To this end select a $z \in \Omega$, and let $\{k_n\}$ be an increasing sequence of positive integers such that $U(x_{k_n}, 2^{-2k_n}) \subset U(z, 2^{-n})$ for $n = 1, 2, \dots$. If

$$\theta_n = \chi_{U(z, 2^{-n})} + 2^{k_n} \chi_{U(x_{k_n}, 2^{-2k_n})},$$

then $\theta_n \in BV_+$, $\|\theta_n\| \leq \pi 2^{2-n}$, and $\pi 2^{-2n} \leq |\theta_n|_1 \leq \pi 2^{1-2n}$. Since

$$F(\theta_n) \geq 2^{k_n} \int_{U(x_{k_n}, 2^{-2k_n})} |x - x_{k_n}|^{-3/2} dx = 4\pi,$$

$x \in S_{\theta_n}$, $d(\theta_n) = 2^{1-n}$, and $r(\theta_n) \geq 1/4$ for $n = 1, 2, \dots$, we see that F is not differentiable at z .

ACKNOWLEDGMENT

The author is obliged to Pertti Mattila for valuable discussions concerning functions of bounded variation.

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