EMBEDDING SUBSPACES OF l_∞^n INTO SPACES WITH SCHAUDER BASIS

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ABSTRACT. It is proved that for sufficiently small $\varepsilon > 0$ and any $0 < \delta < 1/2$, a random n -dimensional subspace E of l_∞^N , where $N = (1 + \varepsilon)n$, has the property: whenever E is embedded into any $(1 + \gamma)n$ -dimensional space with a basis, where $\gamma = c\delta\varepsilon$, then the embedding constant exceeds $c'n^{1/2-\delta}$.

0. INTRODUCTION

In this note we solve, in the negative, the problem raised by Pełczyński in the late 1970s (cf., e.g., [Pe]), particularly, whether every n -dimensional Banach space well-embeds into a $(1 + \varepsilon)n$ -dimensional space with a Schauder basis, for each $\varepsilon > 0$.

It is known that every n -dimensional Banach space is well isomorphic to a well-complemented subspace of an $n^{3/2}$ -dimensional space with a basis [Pe]. It turns out that if one insists on the complementation condition then the above dimension is of the smallest asymptotic order [Ma3, Sz3].

Here we prove that for sufficiently small $\varepsilon > 0$ a random n -dimensional subspace E of l_∞^N , where $N = (1 + \varepsilon)n$, poorly embeds into any $(1 + \gamma)n$ -dimensional space with a basis, with $\gamma = c\varepsilon$. Recently, answering another related question of Pełczyński, it was shown in [JP] that such an E poorly embeds into l_∞^k , for $k \leq N - c \log N$.

We use a standard notation from the theory of Banach spaces (cf., e.g., [Pe]). In particular, the definition of the basis constant $\text{bc}(\cdot)$ can be found there.

1. MAIN RESULTS

Theorem 1. *Let $0 < \delta < 1/2$ and let $\gamma = 2^{-11}\delta$. For every $n \in \mathbb{N}$ there exists an n -dimensional Banach space E such that for any $(1 + \gamma)n$ -dimensional superspace $F \supset E$ and for any $(1 - \gamma)n$ -dimensional subspace $G \subset E$ one has*

$$\text{bc}(F) \geq cn^{1/2-\delta} \quad \text{and} \quad \text{bc}(G) \geq cn^{1/2-\delta},$$

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where $0 < c < 1$ is an absolute constant. Moreover, E can be taken as a subspace of l_∞^N , with $N = (1 + 2^{-6})n$.

As an obvious corollary we get

Corollary 2. Let $0 < \delta < 1/2$ and let $\gamma = 2^{-11}\delta$. Let E be the n -dimensional space from Theorem 1. Then its Banach-Mazur distance to any subspace Z of any $(1 + \gamma)n$ -dimensional Banach space Y with $\text{bc}(Y) = 1$ satisfies

$$d(E, Z) \geq cn^{1/2-\delta}.$$

Theorem 1 easily follows from Theorem 3 below, which is of independent interest. In order to state this theorem we require some notation related to Euclidean structure on $\mathbb{R}^m$. By $\|\cdot\|_2$ we denote the natural l_2 -norm, and by B_2^m , the corresponding unit ball. For a linear operator $T : \mathbb{R}^m \rightarrow \mathbb{R}^m$, by $\|T\|_{2 \rightarrow 2}$ we denote the norm of T as an operator on l_2^m . For $k \geq m$, by $G_{k,m}$ we denote the Grassman manifold of all k -dimensional subspaces in $\mathbb{R}^m$. For $E \in G_{n,m}$ and an operator $T : E \rightarrow E$, T is said to be (k, b) -mixing if there exists a k -dimensional subspace $E_0 \subset E$ such that $\|P_{E_0^+}Tx\|_2 \geq b\|x\|_2$, for all $x \in E_0$. The class of all such operators is denoted by $\text{Mix}_n(k, b)$.

Theorem 3. For any $0 < \delta < 1/2$ and $0 < \varepsilon \leq 2^{-6}$ set $\gamma = 2^{-5}\delta\varepsilon$. For $n \in \mathbb{N}$ set $N = (1 + \varepsilon)n$. There exists an n -dimensional subspace $E \subset l_\infty^N = (\mathbb{R}^N, \|\cdot\|_\infty)$ such that for every operator $T : E \rightarrow E$ with $T \in \text{Mix}_n(3n/8, 1)$ and for every subspace $Z \subset E$ with $\text{codim } Z = \dim E - \dim Z \leq \gamma n$ one has

$$\|T|_Z : Z \rightarrow E\| \geq c(\varepsilon)n^{1/2-\delta},$$

where $c(\varepsilon) > 0$ depends on ε only.

Assuming the validity of Theorem 3 we prove Theorem 1.

Proof of Theorem 1. The proof is formal. Fix $0 < \delta < 1/2$. Let E be an n -dimensional space constructed in Theorem 3 for $\varepsilon = 2^{-6}$. Let $F \supset E$ be any superspace of E with $m = \dim F = (1 + \gamma)n$ and let $\|\cdot\|_2$ be a Euclidean norm on F that extends the natural l_2 -norm from E . Fix $S : F \rightarrow F$ with $S \in \text{Mix}_m(m/2, 1)$. Let $Z \subset E$ be a subspace with $\text{codim } Z \leq \gamma n$ such that $S(Z) \subset E$. There exists $T : E \rightarrow E$ such that $T|_Z = S|_Z$ and $T \in \text{Mix}_n(m/2 - 2\gamma n, 1)$. Thus Theorem 3 implies

$$\|S : F \rightarrow F\| \geq \|S|_Z : Z \rightarrow E\| = \|T|_Z : Z \rightarrow E\| \geq cn^{1/2-\delta}.$$

To complete the lower estimate for $\text{bc}(F)$ recall a folklore fact that for any rank $m/2$ projection $P : F \rightarrow F$ one has $2P \in \text{Mix}_m(m/2, 1)$, with respect to an arbitrary Euclidean norm on F . Indeed, a simple direct computation verifies this for $m = 2$. In the general case, for any rank $m/2$ projection P on l_2^m , the Schmidt decomposition implies that P is a sum of $m/2$ projections of rank 1 acting in mutually orthogonal 2-dimensional subspaces.

The estimate for $\text{bc}(G)$, where G is a subspace of E , is trivial. $\square$

Remarks. 1. The same proof as above shows that whenever $G \subset E$ is a subspace of E with $\dim G \geq (1 - \gamma/2)n$ and $F \supset G$ is a superspace of G with $\dim F \leq (1 + \gamma/2)n$ then $\text{bc}(F) \geq cn^{1/2-\delta}$.

2. A similar argument yields that for every n -dimensional Banach space Y with $\text{bc}(Y) = 1$, the Banach–Mazur distance of every subspace $G \subset E$ with $\dim G = (1 - \gamma/2)n$ to any subspace $Z \subset Y$ admits an estimate

$$d(G, Z) \geq cn^{1/2-\delta}.$$

The same holds if subspaces G and Z are replaced by $(1 + \gamma/2)n$ -dimensional superspaces of E and Y , respectively.

3. Theorem 1 remains valid if the class of Banach spaces with a basis is replaced by some other classes of “regular” Banach spaces such as spaces with enough symmetries or spaces with unique complex structure [Ma1, Ma2, Sz2]. This can be achieved by proving Theorem 3 for operators in $\text{Mix}_n(\alpha n, 1)$, for an appropriate $0 < \alpha < 1$. Therefore, in particular, remarks 1 and 2 hold for these “regular” classes of spaces.

4. Observe that Theorem 3 is equivalent to the fact that for every operator $T : E \rightarrow E$ with $T \in \text{Mix}_n(3n/8, 1)$ its (γn) th Gelfand number is larger than or equal to $cn^{1/2-\delta}$.

2. PROOF OF THEOREM 3

For a Banach space X and a subspace $F \subset X$, let $q_F : X \rightarrow X/F$ denote the canonical quotient map. We will prove Theorem 3 in the dual version, by showing, for $0 < \delta < 1/2$ and $0 < \varepsilon \leq 2^{-6}$, the existence of an n -dimensional quotient X of l_1^N such that for every $T : X \rightarrow X$ with $T \in \text{Mix}_n(3n/8, 1)$ and for every subspace $F \subset X$ with $\dim F \leq \gamma n$ one has

$$(1) \quad \|q_F T : X \rightarrow X/F\| \geq cn^{1/2-\delta}.$$

The space X will be described as $\mathbb{R}^n$ with the unit ball defined below in a probabilistic way. In particular, unless stated otherwise, all operators will be assumed to act from $\mathbb{R}^n$ into itself.

Fix a probability space $(\Omega, \mathbf{P})$. By a standard Gaussian vector in an n -dimensional Euclidean space H_n we mean an H_n -valued random variable with the density

$$(n/2\pi)^{n/2} e^{-\|x\|_2^2/2}$$

with respect to the standard Lebesgue measure in H_n .

Let $g_1, \dots, g_{\varepsilon n}$ be independent standard Gaussian vectors in $\mathbb{R}^n$. For $\omega \in \Omega$, set

$$B_{X(\omega)} = \text{absconv}\{e_1, e_2, \dots, e_n, g_1(\omega), g_2(\omega), \dots, g_{\varepsilon n}(\omega)\},$$

where $e_1, \dots, e_n$ is the standard unit vector basis in $\mathbb{R}^n$. Denote the corresponding norm by $\|\cdot\|_{X(\omega)}$ and set $X(\omega) = (\mathbb{R}^n, \|\cdot\|_{X(\omega)})$. So the unit ball in $X(\omega)$ is $B_{X(\omega)}$.

We will prove the following measure-theoretic version of Theorem 3.

Theorem 4. For any $0 < \delta < 1/2$ and $0 < \varepsilon \leq 2^{-6}$ set $\gamma = 2^{-5}\delta\varepsilon$. There exists $0 < c(\varepsilon) < 1$ such that if $\tilde{\Omega}$ denotes the set

$$\begin{aligned} \tilde{\Omega} = \{ \omega \in \Omega \mid \|q_F T : X(\omega) \rightarrow X(\omega)/F\| \geq c(\varepsilon)n^{1/2-\delta} \\ \text{for every } T \in \text{Mix}_n(3n/8, 1) \text{ and } F \in G_{\gamma n, n} \}, \end{aligned}$$

then $\mathbf{P}(\tilde{\Omega}) \geq 1 - c^n$, where $0 < c < 1$ is an absolute constant.

Remark. As an easy consequence we get that for every $\omega \in \tilde{\Omega}$ and every $T \in \text{Mix}_n(3n/8, 1)$, the cardinality of extreme points of the unit ball, $e \in B_{X(\omega)}$, such that $\|Te\|_{X(\omega)} \geq c(\varepsilon)n^{1/2-\delta}$, is larger than γn .

The proof of the theorem goes along standard lines of reasoning. We establish some measure estimates for subsets of Ω that depend on a fixed operator T , and then we use an ε -net argument to conclude estimates for the set $\tilde{\Omega}$ that is good for all operators T . These two steps are done in two propositions that follow, the first one being the key technical point (rather a delicate one), and the second is a well-known argument.

For any orthogonal projection Q in $\mathbb{R}^n$, $Q(X(\omega))$ is the space $(\ker Q)^\perp$ with the unit ball $Q(B_{X(\omega)})$. Clearly, $Q(X(\omega))$ is isometric to the quotient space $X/\ker Q$.

For every $\omega \in \Omega$ let $E_\omega = \text{span}[g_1(\omega), g_2(\omega), \dots, g_{\varepsilon n}(\omega)]$. If $F \in G_{\gamma n, n}$ with $0 < \gamma < \varepsilon$, let $Q_{\omega, F}$ be the orthogonal projection in $\mathbb{R}^n$ with $\ker Q_{\omega, F} = F + E_\omega$.

Let

$$\Omega_0 = \{\omega \in \Omega \mid 1/2 \leq \|g_i(\omega)\|_2 \leq 2 \text{ for all } i = 1, \dots, \varepsilon n\}.$$

Recall that

$$(2) \quad \mathbf{P}(\Omega_0) \geq 1 - nc_1^n,$$

where $0 < c_1 < 1$ is a universal constant.

Proposition 5. *Let $0 < \alpha < 1$ and $0 < \delta < 1/2$. For an operator $T \in \text{Mix}_n(n/4, 1)$ set*

$$(3) \quad \Omega_T = \{\omega \in \Omega_0 \mid \|Q_{\omega, F} T : X(\omega) \rightarrow Q_{\omega, F}(X(\omega))\| \leq 2\alpha n^{1/2-\delta} \text{ for some } F \in G_{\gamma n, n}\}.$$

Then for every $T \in \text{Mix}_n(n/4, 1)$ one has

$$(4) \quad \mathbf{P}(\Omega_T) \leq C^{n^2} (n/\alpha)^{\gamma n^2} (C_0 \alpha n^{-\delta})^{\varepsilon n^2/16},$$

where $C \geq 1$ and $C_0 \leq 1$ are absolute constants.

The ε -net argument is based on the following fact (cf., e.g., [G, Sz1]), whose proof is omitted.

Proposition 6. *For $A > 0$ let*

$$\mathcal{T}_A = \{T : \mathbb{R}^n \rightarrow \mathbb{R}^n \mid T \in \text{Mix}_n(n/4, 1) \text{ and } \|T : l_1^n \rightarrow l_1^n\| \leq A\}.$$

Then $\mathcal{T}_A$ admits a $(An^{-1/2}/4)$ -net $\mathcal{N}_A$ in the operator norm in l_2^n with the cardinality

$$(5) \quad |\mathcal{N}_A| \leq C_1^{n^2},$$

where $C_1 \geq 1$ is an absolute constant.

Proof of Theorem 4. It is clearly enough to assume that n is sufficiently large. Let C , C_1 , and C_0 be the constants from the above propositions. Set $\alpha = (2C_1 C)^{-32/\varepsilon} C_0^{-2}$ and $A = 2\alpha n^{1/2-\delta}$. Let $\mathcal{N} = \mathcal{N}_A$ be the $(An^{-1/2}/4)$ -net in $\mathcal{T}_A$ constructed in Proposition 6. For $T \in \mathcal{N}$, let Ω_T be defined as in (3). Set

$$\Omega' = \Omega_0 \setminus \bigcup_{T \in \mathcal{N}} \Omega_T.$$

For $n > \alpha^{-2}$ we have, by (4) and (5) and the choice of α and γ ,

$$P(\Omega') \geq (1 - nc_1^n) - (1/2)^{n^2} > 1 - c^n,$$

for some absolute constant $0 < c < 1$. To complete the proof it is enough to show that for every $\omega \in \Omega'$, every $S \in \text{Mix}_n(3n/8, 1)$ and every $F \in G_{\gamma n, n}$, one has

$$(6) \quad \|Q_{\omega, F} S : X(\omega) \rightarrow Q_{\omega, F} X(\omega)\| > \alpha n^{1/2-\delta},$$

which would clearly imply $\tilde{\Omega} \supset \Omega'$.

To this end fix $\omega_0 \in \Omega'$, $S_0 \in \text{Mix}_n(3n/8, 1)$, and $F_0 \in G_{\gamma n, n}$ and write Q for Q_{ω_0, F_0} . Assume to the contrary that

$$(7) \quad \|Q_{\omega_0, F_0} S_0 : X(\omega_0) \rightarrow Q_{\omega_0, F_0} X(\omega_0)\| < \alpha n^{1/2-\delta}.$$

Since $Q(X(\omega)) = Q(l_1^n)$, we also have

$$\|Q S_0 : l_1^n \rightarrow Q(l_1^n)\| < \alpha n^{1/2-\delta}.$$

The lifting property of l_1^n implies that there exists $T_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\|T_0 : l_1^n \rightarrow l_1^n\| < \alpha n^{1/2-\delta}$ and $Q T_0 = Q S_0$. Since $\text{rank}(T_0 - S_0) \leq (\gamma + \varepsilon)n$, we have $T_0 \in \text{Mix}_n(n/4, 1)$; in particular $T_0 \in \mathcal{T}_A$. Pick $T_1 \in \mathcal{N}$ such that $\|T_0 - T_1\|_{2 \rightarrow 2} < (\alpha/2)n^{-\delta}$. By the definition of Ω' we have

$$(8) \quad \|Q T_1 : X(\omega_0) \rightarrow Q(X(\omega_0))\| > 2\alpha n^{1/2-\delta}.$$

By the definition of Ω_0 we have

$$n^{-1/2} B_2^n \subset B_{X(\omega_0)} \subset 2B_2^n,$$

hence, for every $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$,

$$\|Q T : X(\omega_0) \rightarrow Q(X(\omega_0))\| \leq 2n^{1/2} \|T\|_{2 \rightarrow 2}.$$

Therefore, since $Q S_0 = Q T_0$, we get, by (8),

$$\begin{aligned} \|Q S_0 : X(\omega_0) \rightarrow Q(X(\omega_0))\| &\geq \|Q T_1 : X(\omega_0) \rightarrow Q(X(\omega_0))\| \\ &\quad - \|(Q T_1 - Q T_0) : X(\omega_0) \rightarrow Q(X(\omega_0))\| \\ &> 2\alpha n^{1/2-\delta} - \alpha n^{1/2-\delta} = \alpha n^{1/2-\delta}. \end{aligned}$$

This contradicts (7) and completes the proof of the theorem. $\square$

3. PROOF OF PROPOSITION 5

The proof of Proposition 5 is based on two lemmas. The first gives the estimate for a fixed operator $T \in \text{Mix}_n(n/4, 1)$ and a fixed subspace $F \in G_{\gamma n, n}$, while the second one is a reformulation of a known estimate of the cardinality of nets on Grassmann manifolds. Before we are able to state the first lemma we need a simple observation.

Fix $T \in \text{Mix}_n(n/4, 1)$. By the definition of the mixing class, there is $E \subset \mathbb{R}^n$, $\dim E = n/4$ such that $\|P_{E^\perp} T x\|_2 \geq \|x\|_2$ for every $x \in E$. A well-known argument on half-dimensional circular sections of an ellipsoid yields that there exists $E_0 \subset E$ with $\dim E_0 = n/8$ and $\lambda \geq 1$ such that $\|P_{E^\perp} T x\|_2 = \lambda \|x\|_2$ for

every $x \in E_0$. For every $\omega \in \Omega$ and $F \in G_{\gamma n, n}$ let Q_{ω, F, E_0} be the orthogonal projection in $\mathbb{R}^n$ with

$$\ker Q_{\omega, F, E_0} = E_\omega + F + E + P_{E^\perp} T P_{E_0^\perp}(E_\omega).$$

Set

$$(9) \quad \Omega_{T, F} = \{\omega \in \Omega_0 \mid Q_{\omega, F, E_0} T P_{E_0} g_j \in 4\lambda \alpha n^{1/2-\delta} Q_{\omega, F, E_0}(B_{X(\omega)}) \\ \text{for } j = 1, 2, \dots, \varepsilon n\}.$$

Lemma 7. *Let $F \in G_{\gamma n, n}$. Then $\mathbf{P}(\Omega_{T, F}) \leq (C_0 \alpha n^{-\delta})^{\varepsilon n^2/16}$, where $C_0 \geq 1$ is an absolute constant.*

Proof. Set $Q_\omega = Q_{\omega, F, E_0}$ for $\omega \in \Omega$. For $j = 1, 2, \dots, \varepsilon n$ define $g'_j = P_{E_0} g_j$ and $g''_j = P_{E_0^\perp} g_j$. Note that the definition of the Gaussian density clearly implies that random vectors $\{g'_1, g'_2, \dots, g'_{\varepsilon n}, g''_1, g''_2, \dots, g''_{\varepsilon n}\}$ are mutually independent. Hence it follows that Q_ω is independent of the g'_j 's. Indeed, clearly $P_{E^\perp} T P_{E_0^\perp}(E_\omega)$ depends on the distribution of the g''_j 's only and, therefore, is independent of the g'_j 's. Also, $E_\omega + E = E''_\omega + E$, where $E''_\omega = P_{E^\perp}(E_\omega)$ and, hence, also depends on the g''_j 's only. For every fixed $j = 1, 2, \dots, \varepsilon n$ we have

$$\begin{aligned} \{\omega \in \Omega \mid Q_\omega T P_{E_0} g_j \in 4\lambda \alpha n^{1/2-\delta} Q_\omega(B_{X(\omega)})\} \\ = \{\omega \in \Omega \mid Q_\omega T g'_j \in 4\lambda \alpha n^{1/2-\delta} Q_\omega(B_{X(\omega)})\}. \end{aligned}$$

Since $E \subset \ker Q_\omega$, the definition of λ implies that $\lambda^{-1} Q_\omega T|_{E_0}$ is a contraction in l_2^n ; moreover, it has k s -numbers equal to 1, with $k \geq n/8 - \gamma n - 2\varepsilon n \geq n/16$. Hence, using Claim 6.2 in [Sz1] (with $n/3$ replaced by $n/16$) and the fact that $\sqrt{\dim E_0} g'_j$ is a standard Gaussian variable in E_0 (cf., e.g., [Sz1, (3.3)]), we have

$$\begin{aligned} \mathbf{P}(\{\omega \in \Omega \mid Q_\omega T g'_j \in 4\lambda \alpha n^{1/2-\delta} Q_\omega(B_{X(\omega)})\}) \\ \leq \mathbf{P}(\{\omega \in \Omega \mid (\lambda^{-1} Q_\omega T)(\sqrt{\dim E_0} g'_j) \in 16\alpha n^{1/2-\delta} Q_\omega(B_{X(\omega)})\}) \\ \leq (c' 16\alpha n^{1/2-\delta} / n^{1/2})^{n/16} \leq (c''' \alpha n^{-\delta})^{n/16}, \end{aligned}$$

where c' , c'' , and c''' are absolute constants. Hence

$$\mathbf{P}(\Omega_{T, F}) \leq (c''' \alpha n^{-\delta})^{\varepsilon n^2/16},$$

which concludes the proof of the lemma. $\square$

The next lemma is a restatement of Lemma 7.3 in [Sz1].

Lemma 8. *For every $0 < \eta < 1$, the set*

$$\mathcal{P}_{k, n} = \{P : \mathbb{R}^n \rightarrow \mathbb{R}^n \mid P \text{ an orthogonal projection with } \text{rank } P = n - k\}$$

admits an η -net $\mathcal{M}$ in the operator norm in l_2^n with the cardinality $|\mathcal{M}| \leq C n^2 \eta^{-nk}$, where $C > 1$ is an absolute constant.

Proof of Proposition 5. Let $\mathcal{M}$ be an η -net considered in Lemma 8 for $\eta = \alpha n^{-1/2-\delta}/4$ and $k = \gamma n$. We show that for $T \in \text{Mix}_n(n/4, 1)$,

$$(10) \quad \Omega_T \subset \bigcup_{F \in \mathcal{M}} \Omega_{T, F},$$

where $\Omega_{T, F}$ is defined in (9).

To this end fix $\omega_0 \in \Omega_T$ and let $F_0 \in G_{\gamma n, n}$ be such that

$$(11) \quad \|Q_{\omega_0, F_0} T : X(\omega_0) \rightarrow Q_{\omega_0, F_0}(X(\omega_0))\| \leq 2\alpha n^{1/2-\delta}.$$

Pick $F_1 \in \mathcal{M}$ such that $\|P_{F_0} - P_{F_1}\|_{2 \rightarrow 2} < \eta$. To simplify notations write Q_i for Q_{ω_0, F_i, E_0} . Then $\|Q_0 - Q_1\|_{2 \rightarrow 2} < \eta$ as well. Hence for every $y \in \mathbb{R}^n$ we have

$$(12) \quad \|Q_1 Q_0 y - Q_1 y\|_2 < \eta \|y\|_2.$$

In particular, since $\omega_0 \in \Omega_0$, we infer that $n^{-1/2} B_2^n \subset B_{X(\omega_0)} \subset 2B_2^n$. Therefore

$$(13) \quad Q_1 Q_0(B_{X(\omega_0)}) \subset Q_1(B_{X(\omega_0)}) + 2\eta Q_1(B_2^n) \subset \frac{3}{2} Q_1(B_{X(\omega_0)}).$$

For $j = 1, \dots, \varepsilon n$ set $y_j = TP_{E_0} g_j$. Thus $\|y_j\|_2 \leq 2\lambda$. Since $P_{E^\perp} y_j \in \ker Q_i$, for $j = 1, \dots, \varepsilon n$ and $Q_i P_{E^\perp} = Q_i$, for $i = 0, 1$,

$$(14) \quad Q_0 y_j = Q_0 T g_j \quad \text{for } j = 1, \dots, \varepsilon n.$$

Using the latter formula, (11) and (13), we get, for all $j = 1, \dots, \varepsilon n$,

$$\begin{aligned} Q_1 y_j &= Q_1 ((Q_1 - Q_0) y_j + Q_0 T g_j) \\ &\in 2\lambda \eta Q_1(B_2^n) + 2\alpha n^{1/2-\delta} Q_1 Q_0(B_{X(\omega_0)}) \\ &\subset 2\alpha \lambda n^{1/2-\delta} Q_1(B_{X(\omega_0)}), \end{aligned}$$

which combined with the definition of the y_j 's shows that $\omega_0 \in \Omega_{T, F_1}$, completing the proof of (10).

Then the estimate (4) easily follows from Lemmas 7 and 8. $\square$

REFERENCES

- [G] E. D. Gluskin, *The diameter of Minkowski compactum is roughly equal to n* , *Funct. Anal. Appl.* **15** (1981), 72–73.
- [JP] W. B. Johnson and G. Pisier, *The proportional u.a.p. characterizes weak Hilbert spaces*, *J. London Math. Soc.* (to appear).
- [Ma1] P. Mankiewicz, *Finite-dimensional Banach spaces with symmetry constant of order $\sqrt{n}$* , *Studia Math.* **79** (1984), 193–200.
- [Ma2] ———, *Subspace mixing properties of operators in $\mathbb{R}^n$ with applications to Gluskin spaces*, *Studia Math.* **88** (1988), 51–67.
- [Ma3] ———, *Factoring the identity operator on a subspace of l_∞^n* , *Studia Math.* **95** (1989), 133–139.
- [Pe] A. Pełczyński, *Geometry of finite-dimensional Banach spaces and operator ideals*, *Notes in Banach Spaces* (H. E. Lacey, ed.), Univ. of Texas Press, Austin, 1980.
- [Sz1] S. J. Szarek, *The finite dimensional basis problem with an appendix on nets of Grassmann manifolds*, *Acta Math.* **151** (1983), 153–179.
- [Sz2] ———, *On the existence and uniqueness of complex structure and spaces with few operators*, *Trans. Amer. Math. Soc.* **293** (1986), 339–353.
- [Sz3] ———, *A Banach space without a basis which has the bounded approximation property*, *Acta Math.* **159** (1987), 81–98.

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