DENSE PURE SUBGROUPS OF LOCALLY COMPACT GROUPS

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ABSTRACT. We prove that a locally compact abelian (LCA) group has no proper dense pure subgroups if and only if it does not have proper dense subgroups. This solves a problem of Armacost.

The following problem was posed by Dietrich in [2]: does every nondiscrete LCA group have a dense proper subgroup? Rajagopalan and Subrahmanian [7] gave a negative answer to this question. Later, Khan [5] and the author [4] proved that an LCA group G has no proper dense subgroups if and only if $t(G)$ and pG are open in G for each prime p , where $t(G)$ is the maximal torsion subgroup in G and $pG = \{px | x \in G\}$. Among numerous questions in the book of Armacost [1] we mention the following two:

- (i) Is it true that an LCA group G is discrete whenever every pure subgroup of G is closed? (Question 7.24).
- (ii) What are the LCA groups without proper dense pure subgroups? (Question 7.18).

We recall that a subgroup H of an abelian group G is said to be pure in G if $nH = nG \cap H$ for all integers n . Takahashi [8] gave a positive answer to the first question and a partial answer to the second. He proved that a nondiscrete LCA group G contains 2^c proper dense pure subgroups whenever G is compact or $t(G)$ is not open in G . We shall prove that the following holds.

Theorem. *For a nondiscrete LCA group G , the following conditions are equivalent:*

- (i) *the subgroups $t(G)$ and pG are open for all prime p ;*
- (ii) *G has no proper dense subgroups;*
- (iii) *G has no proper dense pure subgroups.*

We begin with three lemmas.

Lemma 1. *Let G be an LCA group, p a prime. If the subgroup pG is not open in G , then for each $a \in G \setminus pG$ there exists a dense subgroup G' of index p in G such that $a \notin G'$.*

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Proof. If there is a subgroup F in G with $pG \subset F$ and $a \in \overline{F} \setminus F$, then any maximal subgroup among those containing F and avoiding a is the required subgroup G' . (Here and in what follows, $\overline{X}$ is the closure of a subset X in G .) To construct F we first choose a nonclosed subgroup H in G containing pG with $a \notin H$. If $a \in \overline{H}$ then $F = H$. Otherwise, $F = gr(a+b, H)$ where $b \in \overline{H} \setminus H$. Indeed, since $\overline{H} \subset \overline{F}$, it follows that $b \in \overline{F}$, and hence $a \in \overline{F}$. On the other hand, if $a \in F$ then $a = k(a+b) + h$, where k is an integer, $h \in H$; i.e., $(1-k)a = kb + h \in \overline{H}$. Since $pG \subset H$ and $a \notin \overline{H}$, we have $1-k \equiv 0 \pmod{p}$, therefore, $kb + h \equiv b + h \equiv 0 \pmod{pG}$, i.e., $b \in H$ —a contradiction. Thus $a \in \overline{F} \setminus F$ and this completes the proof.

Lemma 2. *If G' is a dense subgroup of prime index in an LCA group G and $t(G) \subset G'$, then there exists a subgroup G'' of G such that*

- (i) $G'' \subset G'$;
- (ii) G'' is dense in G ;
- (iii) $G = G'' \oplus A$, where A is a finite cyclic group.

Proof. Let p be the index of G' in G , and a an element of the smallest order among those elements of $t(G)$ not belonging to G' , p^r being the order of a , $r \geq 1$. Let $b = pa$, $A = gr(a)$, $B = gr(b)$. We shall show that B is a pure subgroup of G' . It suffices to check the equality $p^{r-1}G' \cap B = 0$ [3, Proposition 27.1]. Suppose the contrary, $p^{r-1}x = nb$, $n = p^s m$, $0 \leq s < r-1$, m is not divided by p , $x \in G'$. Then $r-s-2 \geq 0$, the element $a' = p^{r-s-2}x - ma$ does not belong to G' , and

$$p^{s+1}a' = p^{r-1}x - nb = 0.$$

Since $s+1 < r$, the order of a' is less than that of a , contradicting the choice of a .

Since B is pure in G' , $G' = G'' \oplus B$ for a subgroup G'' of G' . From this we deduce that $G = G'' \oplus A$. If G'' is not dense in G , then $\overline{G''} = G'' \oplus A'$, where A' is a proper subgroup of A . Hence, $A' \subset B$, whence $\overline{G''} \subset G'$. Since $\overline{G''}$ is a closed subgroup of finite index in G , it follows that $\overline{G''}$ is open, and hence G' is clopen. This contradicts the condition. Thus G'' is dense in G and the proof is complete.

The following lemma is contained in [6, Lemmas 4 and 5]. We provide a slightly different proof.

Lemma 3. *If the subgroup $t(G)$ of torsion elements in an LCA group G is not open, then G has a dense subgroup M such that G/M is isomorphic to the additive group $\mathbb{Q}$ of rationals.*

Proof. First we note that any compact nontorsion group H has a free abelian subgroup of infinite (continuum) rank. This is true if the connected component H of the identity is not trivial. If H is totally disconnected then H contains infinite closed monothetic subgroups for which this fact follows from [1, 5.5(e)].

Let F be a free abelian group of infinite index that is contained in an open compact subgroup H of G . Let D be a subgroup of F with $F/D \simeq \mathbb{Q}$ ($\simeq$ means isomorphism). Then

$$(1) \quad G/D = F/D \oplus M/D$$

for a subgroup M of G . It follows from (1) that $G/M \simeq \mathbb{Q}$. Let $L = H \cap M$. It then follows from (1) that

$$H = F + L \quad \text{and} \quad H/L \simeq F/F \cap L = F/D \simeq \mathbb{Q}.$$

Hence, L is dense in H , i.e., $H \subset \overline{M}$. Since $F \subset H$, it follows that $F \subset \overline{M}$. Now it follows from (1) that $\overline{M} = G$. The proof is complete.

We proceed to the proof of the theorem. The only implication of this theorem that we have to prove is (iii) $\Rightarrow$ (i). By Lemma 3 $t(G)$ is open in G , and by Lemmas 1 and 2 the subgroup pG is open for each prime p .

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