A NOTE ON RIESZ POTENTIALS AND THE FIRST EIGENVALUE

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ABSTRACT. In this paper, we consider the boundedness of Riesz potentials on positively curved manifolds. As an application, we get the greatest lower bound of the essential spectrum of a positively curved manifold.

For a complete Riemannian manifold M of dimension n , the first eigenvalue is defined by the variational formula

$$\mu_0 = \inf_f \frac{\int_M |\nabla f|^2 dx}{\int_M f^2 dx}$$

where f ranges over the smooth functions of compact support, and where ∇ is the gradient of the Riemannian metric of M . If Δ denotes the negative of the Laplace-Beltrami operator of M (and hence is a positive operator) then it is easy to show by spectral theory that μ_0 is the greatest lower bound of the spectrum of M .

As known, if $V(M)$, the volume of M , is finite then $\mu_0 = 0$; if $V(M)$ is infinite then μ_0 is generally different from 0 for negatively curved manifolds (see [2–5])—this fact is very important in analysis. In this paper, we consider the boundedness of Riesz potentials on M ; as an application, we get $\mu_0 = 0$ for positively curved manifolds.

For the Riesz potential $\Delta^{-\alpha/2}$, we know that for $M = R^n$, the Hardy-Littlewood-Sobolev embedding theorem holds, i.e., $\|\Delta^{-\alpha/2}(f)\|_q \leq C_{n,p,q} \|f\|_p$ ($1/q = 1/p - \alpha/n$, $0 < \alpha < n$, $1 < p, q < \infty$), see [7]; for a Cartan-Hadamard manifold M , the Strichartz-Lohoue theorem holds, i.e., $\|\Delta^{-\alpha/2}(f)\|_q \leq C_{n,p} \|f\|_p$ ($1 < p < \infty$, $0 < \alpha < n$), see [6, 8]. But for positively curved manifolds, we have

Theorem. *If $\text{Ric}(M) \geq 0$ and M is noncompact, then $\Delta^{-\alpha/2}$ is $L_p \rightarrow L_q$ bounded iff $V_x(r) \geq C_n r^n$ and $1/q = 1/p - \alpha/n$, where $1 < p, q < \infty$, $0 < \alpha < n$.*

Here $V_x(r)$ is the volume of the geodesic ball with center x and radius r . The theorem shows that for noncompact positively curved manifolds, $\Delta^{-\alpha/2}$ is not bounded in general because $V_x(r) \geq C_n r^n$ is usually not true, for example, for $M = R^1 \times S^{n-1}$.

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Proof of Theorem. We have (see [1, Chapter 1] or [8])

$$\Delta^{-\alpha/2}(f)(x) = \int_M I_\alpha(x, y) f(y) dy,$$

$$I_\alpha(x, y) = \Gamma^{-1} \left(\frac{1}{2} \right) \int_0^\infty s^{\alpha/2} h_s(x, y) s^{-1} ds,$$

where $h_s(x, y)$ is the heat kernel. We know that (see [9])

$$h_s(x, y) \leq C_1 V_x^{-1}(\sqrt{s}) \exp(-c_1 d^2(x, y)/s),$$

$$h_s(x, y) \geq C_2 V_x^{-1}(\sqrt{s}) \exp(-c_2 d^2(x, y)/s),$$

where $d(x, y)$ is the geodesic distance between x and y , and

$$V_x^{-1}(R) \geq C_n(R/r) V_x^{-1}(r), \quad V_x(r) \geq (r/R)^n V_x(R),$$

where $R > r > 0$. So

$$\begin{aligned} I_\alpha(x, y) &\geq C_{n,\alpha} \int_0^\infty s^{\alpha/2} V_x^{-1}(\sqrt{s}) \exp\left(\frac{-c_2 d^2(x, y)}{s}\right) \frac{ds}{s} \\ &\geq C_{n,\alpha} \int_0^{d^2(x, y)} s^{\alpha/2} V_x^{-1}(\sqrt{s}) \exp\left(\frac{-c_2 d^2(x, y)}{s}\right) \frac{ds}{s} \\ &= C_{n,\alpha} V_x^{-1}(d(x, y)) d^\alpha(x, y) \end{aligned}$$

and

$$\begin{aligned} I_\alpha(x, y) &\leq C_{n,\alpha} \int_0^\infty s^{\alpha/2} V_x^{-1}(\sqrt{s}) \exp\left(\frac{-c_1 d^2(x, y)}{s}\right) \frac{ds}{s} \\ &\leq C_{n,\alpha} \left(\int_0^{d^2(x, y)} s^{\alpha/2} V_x^{-1}(\sqrt{s}) \left(\frac{d(x, y)}{\sqrt{s}}\right)^n \exp\left(\frac{-c_1 d^2(x, y)}{s}\right) \frac{ds}{s} \right. \\ &\quad \left. + \int_{d^2(x, y)}^\infty s^{\alpha/2} V_x^{-1}(\sqrt{s}) \left(\frac{d(x, y)}{\sqrt{s}}\right) \exp\left(\frac{-c_1 d^2(x, y)}{s}\right) \frac{ds}{s} \right) \\ &\leq C_{n,\alpha} V_x^{-1}(d(x, y)) d^\alpha(x, y) \end{aligned}$$

for $0 < \alpha < 1$. Now, if $V_x(r) \geq C_n r^n$ and $1/q = 1/p - \alpha/n$, we can easily prove

$$\|\Delta^{-\alpha/2}(f)\|_q \leq C_{n,p} \|f\|_p$$

for $0 < \alpha < 1$ by an easy modification of the proof of Theorem 1 of Chapter V of [7]. Then an iterative process yields the general result for $0 < \alpha < n$. Conversely, if $\Delta^{-\alpha/2}$ is $L_p \rightarrow L_q$ bounded, taking $f = \chi_{B(x_0, r)}$ where x_0 is a fixed point of M , we can get

$$V_{x_0}(r)^{1/p} \geq C_{n,p,q} r^\alpha V_{x_0}(r)^q$$

because for $x \in B(x_0^*, r)$ where x_0^* is a suitable point such that $d(x_0, x_0^*) = 3r$; hence we have $\Delta^{-\alpha/2}(f)(x) \geq C_{n,\alpha} r^\alpha$ and $V_{x_0}(r)/C_n \leq V_{x_0}(r) \leq C_n V_{x_0}(r)$. So

$$V_{x_0}^{1/p-1/q}(r) \geq C_{n,p,q} r^\alpha.$$

Now $V_{x_0}(r) \sim r^n (r \rightarrow 0)$ and $V_{x_0}(r) \lesssim r^n (r \rightarrow \infty)$, thus $1/p - 1/q$ must be α/n and $V_{x_0}(r) \geq C_{n,p,q} r^n$.

Corollary. *If $\text{Ric}(M) \geq 0$ then $\mu_0 = 0$.*

Proof. It is known for $V(M) < \infty$. Now suppose $V(M) = +\infty$, i.e., M is noncompact. If $\mu_0 > 0$, we have $\mu_0^{-1} \|\nabla(f)\|_2 \geq \|f\|_2$. Taking $f = \Delta^{-1/2}(g)$, we get

$$\|\Delta^{-1/2}(g)\|_2 \leq C_n \|\nabla(\Delta^{-1/2})(g)\|_2 \leq C_n \|g\|_2$$

by the L^2 -boundedness of the Riesz transform (see [8]). This means that the Riesz potential $\Delta^{-1/2}$ is L^2 -bounded in contraction to the above theorem.

Since $\mu_0 = 0$ is not an eigenvalue, the above corollary also shows that $\mu_0 = 0$ is just the greatest lower bound of the essential spectrum of M .

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