NEW PROOFS OF WEIGHTED INEQUALITIES FOR THE ONE-SIDED HARDY-LITTLEWOOD MAXIMAL FUNCTIONS

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ABSTRACT. In this note we give a simple proof of the characterization of the weights for which the one-sided Hardy-Littlewood maximal functions apply $L^p(V)$ into weak- $L^p(U)$ and a direct proof of the characterization of the weights for which the one-sided Hardy-Littlewood maximal functions apply $L^p(W)$ into $L^p(W)$.

1. INTRODUCTION AND RESULTS

The one-sided maximal functions M^+f and M^-f of a function $f \in L^1_{\text{loc}}(\mathbb{R})$ have been defined as

$$M^+f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f| \quad \text{and} \quad M^-f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^x |f|.$$

Recently [8, 5], the good weights for these operators have been characterized. In particular the following results were proved.

Theorem 1 [8, 5]. *Let U and V be nonnegative measurable functions. The following are equivalent.*

(a) *There exists a constant $C > 0$ such that for all $\lambda > 0$ and every $f \in L^p(V)$,*

$$\int_{\{x : M^+f(x) > \lambda\}} U \leq \frac{C}{\lambda^p} \int_{\mathbb{R}} |f|^p V.$$

(b) *(U, V) satisfies A_p^+ , i.e., there exists a nonnegative real number A such that*

$$\sup_x \sup_{h>0} \left(\frac{1}{h} \int_{x-h}^x U \right) \left(\frac{1}{h} \int_x^{x+h} V^{-1/(p-1)} \right)^{p-1} = A \quad \text{if } p > 1,$$

$$M^-U(x) \leq AV(x) \quad \text{a.e. if } p = 1.$$

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Theorem 2 [8, 5]. *Let W be a nonnegative measurable function, and let $1 < p < \infty$. The following are equivalent.*

(a) *There exists a constant $C > 0$ such that for every $f \in L^p(W)$,*

$$\int_{\mathbb{R}} (M^+ f)^p W \leq C \int_{\mathbb{R}} |f|^p W.$$

(b) *W , i.e., the pair (W, W) , satisfies A_p^+ .*

The aim of this note is to give a simple proof of Theorem 1 and a direct proof of Theorem 2.

The implication (a) $\Rightarrow$ (b) of Theorem 1 is proved in [8, 5] in the same way as in the classical case of Muckenhoupt A_p classes [7]. In [8] the proof of the other implication is reduced to the corresponding inequality for the Hardy operator [1]. The second section of this note is devoted to giving a simple proof of the implication (b) $\Rightarrow$ (a). Our proof also works for the Hardy operator even though we do not include it explicitly. We thank the referee for pointing out this fact to us.

On the other hand, the implication (a) $\Rightarrow$ (b) of Theorem 2 follows from Theorem 1. Therefore to prove Theorem 2 we only have to show (b) $\Rightarrow$ (a), and for this it will suffice to prove the following proposition.

Proposition 3 [8, 5]. *If $1 < p < \infty$ and W satisfies A_p^+ then there exists s , $1 < s < p$, such that W satisfies A_s^+ .*

It is worth noting that in [8, 5] the implication (b) $\Rightarrow$ (a) of Theorem 2 is proved by using the characterization of the pairs of weights for which M^+ is of strong type (p, p) and the argument of Hunt, Kurtz, and Neugebauer [4]. More precisely, Proposition 3 is not used; however, this proposition is obtained in both papers by using Theorem 2, the analogue of Peter Jones's factorization theorem, and the fact that if W satisfies A_1^+ then $W^{1+\delta}$ also satisfies A_1^+ for some $\delta > 0$. Therefore, the problems of finding a direct proof of Proposition 3 and proving Theorem 2 without using the characterization of the pairs of weights for which M^+ is of strong type (p, p) remain open.

Analogous results hold changing M^+ by M^- and A_p^+ by A_p^- , where A_p^- means that there exists a nonnegative real number A such that

$$\sup_x \sup_{h>0} \left(\frac{1}{h} \int_x^{x+h} U \right) \left(\frac{1}{h} \int_{x-h}^x V^{-1/(p-1)} \right)^{p-1} = A \quad \text{if } p > 1,$$

$$M^+ U(x) \leq AV(x) \quad \text{a.e. if } p = 1.$$

The following sections of the paper are devoted to the proofs of (b) $\Rightarrow$ (a) of Theorem 1 and Proposition 3.

Throughout this paper, χ_E and $|E|$ denote the characteristic function and the Lebesgue measure of the set E , respectively, and $0 \cdot \infty$ is taken to be 0.

2. PROOF OF (b) $\Rightarrow$ (a) IN THEOREM 1

First of all, it is clear that A_p^+ ($p > 1$) is equivalent to saying

$$(2.1) \quad \sup_{a,b} \sup_{a < x < b} \frac{1}{(b-a)^p} \int_a^x U \left(\int_x^b V^{-1/(p-1)} \right)^{p-1} = B < \infty.$$

To prove the implication it will suffice to consider bounded nonnegative functions f with compact support. Let $\lambda > 0$ and $O(\lambda) = \{x : M^+ f(x) > \lambda\}$. It is well known (see [3, pp. 421–424] or [7, Lemma 2.1]) that $O(\lambda) = \bigcup_{j=1}^{\infty} (a_j, b_j)$ where the intervals (a_j, b_j) are bounded, pairwise disjoint, and

$$\lambda \leq \frac{1}{b_j - x} \int_x^{b_j} f \quad \text{for every } x \in [a_j, b_j).$$

Let (a, b) be one of the intervals (a_j, b_j) . It will suffice to show that

$$\int_a^b U \leq \frac{C}{\lambda^p} \int_a^b f^p V,$$

where C is a constant independent of the interval and f .

Let $(a, b) = \bigcup_{k=1}^{\infty} (x_{k-1}, x_k]$ where $\{x_k\}$ is the increasing sequence defined as

$$x_0 = a \quad \text{and} \quad \int_{x_{k-1}}^{x_k} f = \int_{x_k}^b f.$$

It follows that

$$\int_{x_{k-1}}^b f = 4 \int_{x_k}^{x_{k+1}} f.$$

Therefore

$$(2.2) \quad \lambda \leq \frac{1}{b - x_{k-1}} \int_{x_{k-1}}^b f = \frac{4}{b - x_{k-1}} \int_{x_k}^{x_{k+1}} f.$$

Now let $p > 1$. By Hölder's inequality we get

$$\lambda^p \leq \frac{4^p}{(b - x_{k-1})^p} \left(\int_{x_k}^{x_{k+1}} f^p V \right) \left(\int_{x_k}^{x_{k+1}} V^{-1/(p-1)} \right)^{p-1}.$$

Then by A_p^+

$$\begin{aligned} \int_{x_{k-1}}^{x_k} U &\leq \frac{4^p}{\lambda^p (b - x_{k-1})^p} \left(\int_{x_{k-1}}^{x_k} U \right) \left(\int_{x_k}^{x_{k+1}} V^{-1/(p-1)} \right)^{p-1} \int_{x_k}^{x_{k+1}} f^p V \\ &\leq \frac{4^p B}{\lambda^p} \left(\frac{x_{k+1} - x_{k-1}}{b - x_{k-1}} \right)^p \int_{x_k}^{x_{k+1}} f^p V \leq \frac{4^p B}{\lambda^p} \int_{x_k}^{x_{k+1}} f^p V. \end{aligned}$$

Summing over k we get

$$\int_a^b U \leq \frac{4^p B}{\lambda^p} \int_{x_1}^b f^p V \leq \frac{4^p B}{\lambda^p} \int_a^b f^p V.$$

If $p = 1$ we simply observe that by (2.2) and A_1^+ we have

$$\begin{aligned} \int_{x_{k-1}}^{x_k} U &\leq \frac{4}{\lambda(b - x_{k-1})} \int_{x_k}^{x_{k+1}} f \int_{x_{k-1}}^{x_k} U \\ &\leq \frac{4}{\lambda} \int_{x_k}^{x_{k+1}} f(t) \left(\frac{1}{t - x_{k-1}} \int_{x_{k-1}}^t U \right) dt \leq \frac{4A}{\lambda} \int_{x_k}^{x_{k+1}} f V, \end{aligned}$$

and now summing over k we get the result as in the case $p > 1$.

3. PROOF OF PROPOSITION 3

The proof follows the pattern of the corresponding proof in [2] for Muckenhoupt's A_p classes. To prove the proposition we will need the following lemmas.

Lemma 4. *Let $1 < p < \infty$ and $\lambda > 0$. If W satisfies A_p^+ and (a, b) is a bounded interval such that W is integrable on (a, b) and*

$$\lambda \leq \frac{1}{x-a} \int_a^x W \quad \text{for every } x \in (a, b],$$

then there exist positive numbers α and β , independent of λ and the interval, such that

$$|\{x \in (a, b) : W(x) > \beta\lambda\}| > \alpha(b-a).$$

Lemma 5 (a weak reverse Hölder's inequality). *Let $1 < p < \infty$. If W satisfies A_p^+ then there exist positive numbers δ and C such that*

$$\int_a^b W^{1+\delta} \leq C \int_a^b W (M^-(W\chi_{(a,b)})(b))^\delta$$

for every bounded interval (a, b) , and therefore

$$M^-(W^{1+\delta}\chi_{(a,b)})(b) \leq C(M^-(W\chi_{(a,b)})(b))^{1+\delta}.$$

The constant C depends only on δ and the constant of the A_p^+ condition.

Proof of Lemma 4. Let $\{x_k\}_{k=0}^{-\infty}$ be the sequence in the interval $(a, b]$ defined as

$$x_0 = b, \quad x_k < x_{k+1}, \quad \text{and} \quad \int_{x_k}^{x_{k+1}} W = \int_a^{x_k} W, \quad k \leq -1.$$

Then

$$|\{x \in (a, b) : W(x) \leq \beta\lambda\}| = \sum_{k=-1}^{-\infty} |\{x \in [x_k, x_{k+1}) : W(x) \leq \beta\lambda\}|.$$

The assumptions in the lemma and the definition of the sequence give

$$\lambda \leq \frac{1}{x_{k+1}-a} \int_a^{x_{k+1}} W = \frac{2}{x_{k+1}-a} \int_a^{x_k} W = \frac{4}{x_{k+1}-a} \int_{x_{k-1}}^{x_k} W.$$

Therefore if $E_k = \{x \in [x_k, x_{k+1}) : W(x) \leq \frac{4\beta}{x_{k+1}-a} \int_{x_{k-1}}^{x_k} W\}$ we have

$$(3.1) \quad |\{x \in (a, b) : W(x) \leq \beta\lambda\}| \leq \sum_{k=-1}^{-\infty} |E_k|.$$

On the other hand, the definition of E_k and condition A_p^+ give

$$\begin{aligned} & \frac{1}{\beta} \left(\frac{|E_k|}{x_{k+1} - x_{k-1}} \right)^{p-1} \\ &= \frac{4}{x_{k+1} - a} \int_{x_{k-1}}^{x_k} W \left(\frac{1}{x_{k+1} - x_{k-1}} \int_{E_k} \left(\frac{4\beta}{x_{k+1} - a} \int_{x_{k-1}}^{x_k} W \right)^{-1/(p-1)} \right)^{p-1} \\ &\leq \frac{4}{x_{k+1} - x_{k-1}} \int_{x_{k-1}}^{x_k} W \left(\frac{1}{x_{k+1} - x_{k-1}} \int_{x_k}^{x_{k+1}} W^{-1/(p-1)} \right)^{p-1} \leq 4B. \end{aligned}$$

Now, this inequality together with (3.1) gives

$$\begin{aligned} |\{x \in (a, b) : W(x) \leq \beta\lambda\}| &\leq (4B\beta)^{1/(p-1)} \sum_{k=-1}^{-\infty} (x_{k+1} - x_{k-1}) \\ &\leq 2(4B\beta)^{1/(p-1)}(b-a). \end{aligned}$$

Therefore

$$(b-a) - |\{x \in (a, b) : W(x) > \beta\lambda\}| \leq 2(4B\beta)^{1/(p-1)}(b-a),$$

and then

$$|\{x \in (a, b) : W(x) > \beta\lambda\}| \geq (b-a)(1 - 2(4B\beta)^{1/(p-1)}).$$

Taking β small enough we are done.

Proof of Lemma 5. Let $\lambda_0 = M^-(W\chi_{(a,b)})(b)$. If $\lambda_0 = \infty$ then the inequality in Lemma 5 holds for all positive numbers C and δ . Assume $\lambda_0 < \infty$, and let $\lambda > M^-(W\chi_{(a,b)})(b) = \lambda_0$. If $O(\lambda) = \{x : M^-(W\chi_{(a,b)})(x) > \lambda\}$ then $O(\lambda) = \bigcup_{j=1}^{\infty} (a_j, b_j)$ where the intervals (a_j, b_j) are bounded, pairwise disjoint, included in (a, b) , and

$$\lambda = \frac{1}{b_j - a_j} \int_{a_j}^{b_j} W \leq \frac{1}{x - a_j} \int_{a_j}^x W \quad \text{for every } x \in (a_j, b_j).$$

Then we have by Lemma 4

$$\begin{aligned} \int_{\{x \in (a, b) : W(x) > \lambda\}} W &\leq \sum_{j=1}^{\infty} \int_{a_j}^{b_j} W = \lambda \sum_{j=1}^{\infty} (b_j - a_j) \\ &\leq \frac{\lambda}{\alpha} \sum_{j=1}^{\infty} |\{x \in (a_j, b_j) : W(x) > \beta\lambda\}| \\ &\leq \frac{\lambda}{\alpha} |\{x \in (a, b) : W(x) > \beta\lambda\}|. \end{aligned}$$

Multiplying by $\lambda^{\delta-1}$ and integrating from λ_0 to ∞ we get

$$\begin{aligned} \int_{\lambda_0}^{\infty} \lambda^{\delta-1} \int_{\{x \in (a, b) : W(x) > \lambda\}} W(x) dx d\lambda &\leq \int_{\lambda_0}^{\infty} \frac{\lambda^{\delta}}{\alpha} |\{x \in (a, b) : W(x) > \beta\lambda\}| d\lambda \\ &\leq \int_0^{\infty} \frac{\lambda^{\delta}}{\alpha} |\{x \in (a, b) : W(x) > \beta\lambda\}| d\lambda = \frac{1}{(1+\delta)\alpha\beta^{1+\delta}} \int_a^b W^{1+\delta}. \end{aligned}$$

On the other hand, the left-hand side of (3.2) is equal to

$$\begin{aligned}
 (3.2) \quad & \int_{\{x \in (a, b) : W(x) > \lambda_0\}} W(x) \int_{\lambda_0}^{W(x)} \lambda^{\delta-1} d\lambda dx \\
 &= \int_{\{x \in (a, b) : W(x) > \lambda_0\}} W(x) \left(\frac{W^\delta(x)}{\delta} - \frac{\lambda_0^\delta}{\delta} \right) dx \\
 &\geq \frac{1}{\delta} \int_a^b W^{1+\delta}(x) dx - \frac{\lambda_0^\delta}{\delta} \int_a^b W(x) dx.
 \end{aligned}$$

This inequality together with (3.2) gives

$$\left(\frac{1}{\delta} - \frac{1}{(1+\delta)\alpha\beta^{1+\delta}} \right) \int_a^b W^{1+\delta} \leq \frac{\lambda_0^\delta}{\delta} \int_a^b W.$$

The proof of Lemma 5 is finished by taking δ small enough.

Proof of Proposition 3. We may assume without loss of generality that $W > 0$ a.e. Actually, if $E = \{x : W = 0 \text{ a.e. in } (-\infty, x)\}$ and α is the supremum of E then the A_p^+ condition implies $\int_a^b W^{-1/(p-1)} < \infty$ for all a, b with $\alpha < a < b$, and therefore $W > 0$ a.e. in (α, ∞) and $W = 0$ a.e. in $(-\infty, \alpha]$. Consequently, if we want to prove that W satisfies A_s^+ it will be enough to show (see (2.1))

$$\sup_{\alpha < a, b} \sup_{a < x < b} \frac{1}{(b-a)^s} \int_a^x W \left(\int_x^b W^{-1/(s-1)} \right)^{s-1} < \infty,$$

i.e., we only have to work in the set where W is positive a.e.

Assume $W > 0$ a.e. First, we observe that the A_p^+ condition implies that $\sigma = W^{-1/(p-1)}$ is locally integrable. Second, we note that W satisfies A_p^+ if and only if σ satisfies A_q^- where $p+q = pq$. Then by the analogue of Lemma 5 for A_q^- classes, we have that there exist $\delta > 0$ and $C > 0$ such that for every bounded interval (x, b)

$$(3.3) \quad M^+(\sigma^{1+\delta} \chi_{(x, b)})(x) \leq C(M^+(\sigma \chi_{(x, b)})(x))^{1+\delta}.$$

(In this proof, the letter C will always mean a positive constant not necessarily the same at each occurrence.)

Now we will show that W satisfies A_s^+ , where $s = \frac{p+\delta}{1+\delta}$, by proving that (2.1) holds with s instead of p and $U = V = W$.

Fix $a < x < b$. Since σ is locally integrable, it follows from (3.3) that the same holds for $\sigma^{1+\delta}$. Therefore, there exists a finite decreasing sequence $x_0 = x > x_1 > \dots > x_N \geq a = x_{N+1}$ such that

$$(3.4) \quad \int_{x_k}^b \sigma^{1+\delta} = 2^k \int_x^b \sigma^{1+\delta} \quad \text{if } k = 0, \dots, N \quad \text{and} \quad \int_a^{x_N} \sigma^{1+\delta} < 2^N \int_x^b \sigma^{1+\delta}.$$

From (3.4) easily follows the fact that for every $k = 0, \dots, N$

$$(3.5) \quad \int_{x_{k+1}}^b \sigma^{1+\delta} \leq 2^{k+1} \int_x^b \sigma^{1+\delta},$$

which will be useful later on. On the other hand, by (3.4) and (3.3)

$$\begin{aligned} \int_a^x W \left(\frac{1}{b-a} \int_x^b \sigma^{1+\delta} \right)^s &= \sum_{k=0}^N \frac{1}{2^{ks}} \int_{x_{k+1}}^{x_k} W \left(\frac{1}{b-a} \int_{x_k}^b \sigma^{1+\delta} \right)^s \\ &\leq \sum_{k=0}^N \frac{1}{2^{ks}} \int_{x_{k+1}}^{x_k} W(y) (M^+(\sigma^{1+\delta} \chi_{(y,b)})(y))^s dy \\ &\leq \sum_{k=0}^N \frac{C}{2^{ks}} \int_{x_{k+1}}^{x_k} W(y) (M^+(\sigma \chi_{(x_{k+1},b)})(y))^{p+\delta} dy. \end{aligned}$$

Since W satisfies A_p^+ , we know by Theorem 1 that M^+ applies $L^p(W)$ into weak- $L^p(W)$. Then, by Marcinkiewicz's interpolation theorem, M^+ applies $L^{p+\delta}(W)$ into $L^{p+\delta}(W)$. This together with (3.5) and the fact that $s > 1$ gives

$$\begin{aligned} \int_a^x W \left(\frac{1}{b-a} \int_x^b \sigma^{1+\delta} \right)^s &\leq C \sum_{k=0}^N \frac{1}{2^{ks}} \int_{x_{k+1}}^b \sigma^{1+\delta} \\ &\leq C \sum_{k=0}^N \frac{2^{k+1}}{2^{ks}} \int_x^b \sigma^{1+\delta} \leq C \int_x^b \sigma^{1+\delta} < \infty. \end{aligned}$$

Then (2.1), with s instead of p and $U = V = W$, follows from this inequality, and therefore the proof of Proposition 3 is finished.

4. FINAL REMARKS

1. The corresponding proofs work for the operators M_g^+ , M_g^- studied in [5] and defined as

$$M_g^+ f(x) = \sup_{h>0} \frac{\int_x^{x+h} |f|g}{\int_x^{x+h} g}, \quad M_g^- f(x) = \sup_{h>0} \frac{\int_{x-h}^x |f|g}{\int_{x-h}^x g},$$

where g is a positive locally integrable function.

2. The good pairs of weights for the strong type inequalities were characterized in [8] and with a different and simpler proof in [5]. A new simplification in a more general context will appear in [6].

3. P. Ortega Salvador uses the argument of this note to prove weighted inequalities on Orlicz spaces and $L_{p,q}$ -spaces. His proofs will appear elsewhere.

4. It is easy to see that the weights in A_p^+ classes do not satisfy the reverse Hölder's inequality. An example is $W(x) = e^x$.

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