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ON ISOLATED POINTS OF THE SPECTRUM OF A BOUNDED LINEAR OPERATOR

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ABSTRACT. For a bounded linear operator A on a Banach space we characterize the isolated points in the spectrum of A, the Riesz points of A, and the poles of the resolvent of A.

1. TERMINOLOGY AND INTRODUCTION

Throughout this paper E will be an infinite-dimensional complex Banach space and A will be a bounded linear operator on E. We denote by $N(A)$ the kernel and by $A(E)$ the range of A. The spectrum of A will be denoted by $\sigma(A)$. The resolvent set $\varrho(A)$ of A is the complement of $\sigma(A)$ in the complex plane $\mathbb{C}$. For any λ in $\varrho(A)$ the resolvent operator $(\lambda I - A)^{-1}$ is denoted by $R_\lambda(A)$.

Let λ_0 be an isolated point in $\sigma(A)$. The spectral projection corresponding to λ_0 will be denoted by P_{λ_0}. We have $E = P_{\lambda_0}(E) \oplus N(P_{\lambda_0})$.

In [3] Mbekhta introduced two important subspaces of E:

$$K(A) = \{x \in E : \text{ there exist } c > 0 \text{ and a sequence } (x_n)_{n\geq 1} \subseteq E \text{ such that } Ax_1 = x,\ Ax_{n+1} = x_n \text{ for all } n \in \mathbb{N}, \text{ and } \|x_n\| \leq c^n\|x\| \text{ for all } n \in \mathbb{N}\},$$

$$H_0(A) = \left\{x \in E : \lim_{n\to\infty} \|A^n x\|^{1/n} = 0\right\}$$

and proved the following

Theorem 1. *A point $\lambda_0 \in \sigma(A)$ is isolated in $\sigma(A)$ if and only if there is a bounded projection P on E such that*

$$P(E) = H_0(\lambda_0 I - A) \quad \textit{and} \quad N(P) = K(\lambda_0 I - A).$$

In the present paper we shall prove that $\lambda_0 \in \sigma(A)$ is an isolated point of $\sigma(A)$ if and only if $K(\lambda_0 I - A)$ is closed and $E = K(\lambda_0 I - A) \oplus H_0(\lambda_0 I - A)$ (where $\oplus$ denotes the algebraically direct sum). This characterization leads to

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a characterization of the poles of the resolvent of A and to a characterization of the Riesz points of A. This will be done in §3 of this paper.

2. Preliminary results

The operator A is said to have the *single-valued extension property* (SVEP) in $\lambda_0 \in \mathbb{C}$ if for any holomorphic function $f: U \to E$, where U is a neighbourhood of λ_0, with $(\lambda I - A)f(\lambda) \equiv 0$, the result is $f(\lambda) \equiv 0$. We say that A has the SVEP if A has the SVEP in each $\lambda \in \mathbb{C}$.

The following theorem collects some results due to Mbekhta (see [4]).

Theorem 2. (a) $A(K(A)) = K(A)$ *and* $A(H_0(A)) \subseteq H_0(A)$;
(b) A *has the SVEP in* λ_0 *if* $H_0(\lambda_0 I - A)$ *is closed;*
(c) A *has the SVEP in* λ_0 *if and only if* $K(\lambda_0 I - A) \cap H_0(\lambda_0 I - A) = \{0\}$.

The proof of the next result is immediate.

Proposition 1. *Let* $x \in H_0(A)$ *and define the function* g *on* $\mathbb{C}\backslash\{0\}$ *by*

$$g(\lambda) = \sum_{n=0}^{\infty} \frac{A^n x}{\lambda^{n+1}}.$$

Then g *is holomorphic and* $(\lambda I - A)g(\lambda) = x$ *for all* $\lambda \in \mathbb{C}\backslash\{0\}$.

Proposition 2. *Let* F *be a closed subspace of* E *such that* $A(F) = F$. *Then* $F \subseteq K(A)$.

Proof. Since F is a Banach space and $A(F) = F$, the open mapping theorem shows the existence of a constant $c > 0$ so that

$$\text{for each } u \in F \text{ there exists } v \in F \text{ such that} \quad Av = u \quad \text{and} \quad \|v\| \leq c\|u\|. \tag{2.1}$$

Let $x \in F$. Use (2.1) to construct a sequence $(x_n)_{n\geq 1} \subseteq F$ such that $Ax_1 = x$, $Ax_{n+1} = x_n$, and $\|x_n\| \leq c^n \|x\|$. It follows that $x \in K(A)$. □

Let us review the classical definitions of ascent and descent. The *ascent* $p(A)$ and the *descent* $q(A)$ are the extended integers given by

$$p(A) = \inf\{n \geq 0 : N(A^n) = N(A^{n+1})\},$$
$$q(A) = \inf\{n \geq 0 : A^n(E) = A^{n+1}(E)\}.$$

The infimum over the empty set is taken to be ∞. It follows from [2, Satz 72.3] that if $p(A)$ and $q(A)$ are both finite then they are equal.

We have the following characterization of the poles of the resolvent of A (see [2, Satz 101.2]):

Theorem 3. *The complex number* λ_0 *is a pole of* $R_\lambda(A)$ *if and only if* $0 < p(\lambda_0 I - A) = q(\lambda_0 I - A) < \infty$. *In this case we have*

$$P_{\lambda_0}(E) = N((\lambda_0 I - A)^p) \quad \text{and} \quad N(P_{\lambda_0}) = (\lambda_0 I - A)^p(E),$$

where $p = p(\lambda_0 I - A)$ *is the order of the pole* λ_0.

The next proposition is a generalization of [1, Theorem 2].

Proposition 3. *Suppose that A has the SVEP in $\lambda_0 = 0$ and $q(A) < \infty$. Then $p(A) = q(A)$.*

Proof. Let $q = q(A)$, $B = A^q$, and $\widehat{E} = E/N(B)$. Since $N(B)$ is closed, $\widehat{E}$ is a Banach space. Let $\widehat{B}\colon \widehat{E} \to E$ be the corresponding canonical injection. It is easy to see that the operator $\widehat{B}^{-1}\colon A^q(E) \to \widehat{E}$ is closed, thus $A^q(E)$ is the domain of a closed linear operator. Since $A(A^q(E)) = A^q(E)$ and A has the SVEP in 0, [1, Corollary 4] shows that $N(A) \cap A^q(E) = \{0\}$. Use [2, Satz 72.1] to derive $p(A) < \infty$. $\square$

Corollary 1. *The following assertions are equivalent:*

(a) 0 *is a pole of* $R_\lambda(A)$;
(b) A *has the SVEP in* 0 *and* $q(A) < \infty$.

Proof. (a) implies (b). Since 0 is isolated in $\sigma(A)$, A has the SVEP in 0. Theorem 3 shows that $q(A) < \infty$.

(b) implies (a). Proposition 3 and Theorem 3. $\square$

3. Isolated points of the spectrum

The starting point of our investigation is

Proposition 4. *Suppose that* 0 *is an isolated point in* $\sigma(A)$. *Then*

(a) $P_0(E) = H_0(A)$;
(b) $N(P_0) = K(A)$.

Proof. (a) follows from [2, Satz 100.2].

(b) Since 0 is isolated in $\sigma(A)$, $\sigma(A_{|P_0(E)}) = \{0\}$ and $0 \in \varrho(A_{|N(P_0)})$ [2, Satz 100.1]. Then $N(P_0)$ is closed and $A(N(P_0)) = N(P_0)$. Hence, by Proposition 2, $N(P_0) \subseteq K(A)$. By Theorem 2(c), $K(A) \cap H_0(A) = \{0\}$. Therefore,

$$\begin{aligned} K(A) &= K(A) \cap E = K(A) \cap [N(P_0) \oplus P_0(E)] \\ &= N(P_0) + K(A) \cap H_0(A) = N(P_0). \quad \square \end{aligned}$$

Theorem 4. *The following assertions are equivalent:*

(a) 0 *is an isolated point in* $\sigma(A)$;
(b) $K(A)$ *is closed and* $E = K(A) \oplus H_0(A)$ ($\oplus$ *denotes the algebraically direct sum*).

Proof. (a) implies (b). Use Proposition 4 or Theorem 1.

(b) implies (a). Since $K(A)$ is closed, $A(K(A)) = K(A)$ (Theorem 2(a)), and $N(A) \subseteq H_0(A)$, the operator $A\colon K(A) \to K(A)$ is invertible. Hence there exists $\varepsilon > 0$ such that $\lambda I - A_{|K(A)}$ is invertible if $|\lambda| < \varepsilon$. In particular,

$$(\lambda I - A)(K(A)) = K(A) \quad \text{if } |\lambda| < \varepsilon. \tag{3.1}$$

Since for all $\lambda \neq 0$, $N(\lambda I - A) \subseteq K(A)$, we have

$$N(\lambda I - A) = \{0\} \quad \text{if } 0 < |\lambda| < \varepsilon. \tag{3.2}$$

By Proposition 1, for all $\lambda \neq 0$,

$$H_0(A) \subseteq (\lambda I - A)(E). \tag{3.3}$$

Now, (3.1) and (3.3) imply

$$E = K(A) \oplus H_0(A) \subseteq (\lambda I - A)(E) \quad \text{if } 0 < |\lambda| < \varepsilon.$$

Consequently, $\{\lambda \in \mathbb{C} : 0 < |\lambda| < \varepsilon\} \subseteq \varrho(A)$ and the proof is complete. □

Now we are in a position to present the announced characterization of the poles of the resolvent of A.

Theorem 5. *The following assertions are equivalent*:

(a) 0 *is a pole of the resolvent of* A;
(b) A *has the SVEP in* 0 *and* $q(A) < \infty$;
(c) *There exists* $p \in \mathbb{N}$ *such that*

$$N(A^p) = H_0(A) \quad \textit{and} \quad A^p(E) = K(A);$$

(d) A *has the SVEP in* 0 *and there exists* $p \in \mathbb{N}$ *such that* $K(A) = A^p(E)$;
(e) $q(A) < \infty$ *and* $H_0(A)$ *is closed.*

Proof. By Corollary 1, (a) and (b) are equivalent.

(a) implies (c). Use Theorem 3 and Proposition 4.

(c) implies (a). By Theorem 3, we have to show that $p(A)$ and $q(A)$ are both finite. Since

$$N(A^{p+1}) \subseteq H_0(A) = N(A^p) \subseteq N(A^{p+1}),$$

we have $p(A) \leq p$. Use Theorem 2(a) to derive $A^{p+1}(E) = A(A^p(E)) = A(K(A)) = K(A) = A^p(E)$. Thus $q(A) \leq p$.

(a) implies (d). Use (b) and (c).

(d) implies (b). As in the proof of "(c) implies (a)," we have $A^p(E) = A^{p+1}(E)$, hence $q(A) < \infty$.

(a) implies (e). Clear.

(e) implies (b). By Theorem 2(b), A has the SVEP in 0. □

The remainder of this paper deals with Riesz points and Riesz operators. A complex number λ_0 is called a *Riesz point* of A, if

$$p(\lambda_0 I - A) = q(\lambda_0 I - A) < \infty \quad \text{and} \quad \dim N(\lambda_0 I - A) = \operatorname{codim}(\lambda_0 I - A)(E) < \infty.$$

Note that a Riesz point of A is either a pole of the resolvent (and hence isolated in $\sigma(A)$) or a point in the resolvent set $\varrho(A)$.

Proposition 5. *The complex number* $\lambda_0 \in \sigma(A)$ *is a Riesz point of* A *if and only if* λ_0 *is isolated in* $\sigma(A)$ *and the corresponding spectral projection is finite dimensional.*

Proof. [2, Satz 105.3]. □

The next theorem uses the subspaces $K(A)$ and $H_0(A)$ and the SVEP to characterize the Riesz points of A.

Theorem 6. *The following assertions are equivalent*:

(a) 0 *is a Riesz point of* A;
(b) $K(A)$ *is closed,* $\dim H_0(A) < \infty$, *and* $E = K(A) \oplus H_0(A)$, *where* $\oplus$ *denotes the algebraically direct sum*;
(c) $q(A) < \infty$ *and* $\dim H_0(A) < \infty$;
(d) $\dim H_0(A) < \infty$ *and* $K(A) = A^P(E)$ *for some* $p \in \mathbb{N}$;
(e) A *has the SVEP in* 0, $q(A) < \infty$, *and* $\dim N(A) < \infty$.

Proof. (a) $\Leftrightarrow$ (b). Proposition 4 and Theorem 4.

(c) $\Rightarrow$ (a). Since $N(A^n) \subseteq N(A^{n+1}) \subseteq H_0(A)$ and $\dim H_0(A) < \infty$, there exists $p \in \mathbb{N}$ such that $\dim N(A^p) = \dim N(A^{p+1}) < \infty$. This gives $N(A^p) = N(A^{p+1})$, thus $p(A) < \infty$. By Theorem 3, 0 is a pole of $R_\lambda(A)$, hence 0 is isolated in $\sigma(A)$. Proposition 4 shows that $\dim P_0(E) < \infty$. Now use Proposition 5.

(a) $\Rightarrow$ (d). Propositions 4 and 5 show that $\dim P_0(E) = \dim H_0(A) < \infty$ and $N(P_0) = K(A)$. Since 0 is a pole of $R_\lambda(A)$, we conclude from Theorem 3 that $K(A) = A^p(E)$ for some $p \in \mathbb{N}$.

(d) $\Rightarrow$ (c). $A^{p+1}(E) = A(K(A)) = K(A) = A^p(E)$, thus $q(A) < \infty$.

(a) $\Rightarrow$ (e). Clear.

(e) $\Rightarrow$ (a). By Proposition 3, $p(A) = q(A) < \infty$. [2, Satz 72.6] shows that $\dim N(A) = \operatorname{codim} A(E) < \infty$, thus 0 is a Riesz point of A. $\square$

The operator A is called a *Riesz operator* if every $\lambda \in \sigma(A)\backslash\{0\}$ is a Riesz point of A.

An immediate consequence of Theorem 6 is

Theorem 7. *The following assertions are equivalent*:

(a) *A is a Riesz operator*;
(b) $\dim H_0(\lambda I - A) < \infty$, $E = K(\lambda I - A) \oplus H_0(\lambda I - A)$, *and* $K(\lambda I - A)$ *is closed for all* $\lambda \in \sigma(A)\backslash\{0\}$;
(c) $q(\lambda I - A) < \infty$ *and* $\dim H_0(\lambda I - A) < \infty$ *for all* $\lambda \in \sigma(A)\backslash\{0\}$;
(d) $\dim H_0(\lambda I - A) < \infty$ *for all* $\lambda \in \sigma(A)\backslash\{0\}$ *and for each* $\lambda \in \sigma(A)\backslash\{0\}$ *there exists* $p(\lambda) \in \mathbb{N}$ *such that* $K(\lambda I - A) = (\lambda I - A)^{p(\lambda)}(E)$.

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