PERIODIC L^2 -SOLUTIONS OF AN INTEGRODIFFERENTIAL EQUATION IN A HILBERT SPACE

OLOF J. STAFFANS

(Communicated by J. Marshall Ash)

ABSTRACT. Let A be a closed, densely defined operator in a Hilbert space X , and let μ , ν , and η be finite, scalar-valued measures on $\mathbf{R}$. Consider the abstract integrodifferential equation

$$\int_{\mathbf{R}} \frac{d}{dt} u(t-s) \mu(ds) + \int_{\mathbf{R}} u(t-s) \nu(ds) + \int_{\mathbf{R}} Au(t-s) \eta(ds) = f(t), \quad t \in \mathbf{R},$$

where f is a 2π -periodic L^2 function with values in X . We give necessary and sufficient conditions for this equation to have a mild 2π -periodic L^2 -solution with values in X for all f , as well as necessary and sufficient conditions for it to have a strong solution for all f . Furthermore, we give necessary and sufficient conditions for the operator mapping f into the periodic solution u to be compact. These results are applied to prove existence of periodic solutions of a nonlinear equation.

1. INTRODUCTION

Let A be a closed, densely defined operator in a Hilbert space X , and let μ , ν , and η be finite, scalar-valued measures on $\mathbf{R}$. Consider the abstract integrodifferential equation

$$(1.1) \quad \mu * u' + \nu * u + \eta * Au = f,$$

where u' stands for the time-derivative of u and f is a 2π -periodic L^2 -function with values in X . Here $\mu * u'$ represents the function $(\mu * u')(t) = \int_{\mathbf{R}} u'(t-s) \mu(ds)$, and $\nu * u$ and $\eta * Au$ are defined analogously. We give necessary and sufficient conditions for this equation to have a mild 2π -periodic L^2 -solution with values in X for all f ; the definition of a mild solution is given in Definition 2.3 below. We also give necessary and sufficient conditions for it to have a strong solution with values in X for all f ; a strong solution means one that belongs locally to $W^{1,2}(X) \cap L^2(\mathcal{D}(A))$, where $\mathcal{D}(A)$ is the domain of A . Furthermore, we give necessary and sufficient conditions for the operator mapping f into the periodic solution u to be compact. This is important

Received by the editors August 6, 1990 and, in revised form, June 26, 1991.

1991 *Mathematics Subject Classification.* Primary 42A45, 34K30, 45K05, 35B10, 47G05.

Key words and phrases. Integrodifferential equation, well-posedness, periodic solutions, L^2 -multipliers, compact solution operator.

when one wants to prove perturbation results where f has been replaced by a nonlinear function $f(u)$ of u .

In §5 these results are applied to a nonlinear perturbation of (1.1). This leads to a significant simplification and improvement of [1].

The questions studied here are solely related to existence, uniqueness, and continuous dependence of periodic solutions. Nothing is said about the stability or instability of such solutions. Some statements concerning a weak type of stability of periodic solutions could be obtained if one combined the results presented here with those given in Staffans [4, Chapter 11], where the same problem is studied in a global L^2 -setting.

Extensions of the results presented here to higher order convolution equations are given in Staffans [5].

In the sequel we shall use the following notation.

- $\mathcal{T}$ The real line $\mathbf{R}$ where the points $t + 2\pi m$, $m \in \mathbf{Z}$, are identified.
- $L^2(\mathcal{T}; X)$ The set of 2π -periodic functions that are locally in L^2 with values in the Hilbert space X . This is in a Hilbert space, and one can choose $\langle u, v \rangle_{L^2(\mathcal{T}; X)} = \frac{1}{2\pi} \int_0^{2\pi} \langle u(t), v(t) \rangle_X dt$ to be the inner product.
- $H^\alpha(\mathcal{T}; X)$ Here $\alpha \geq 0$. The set of $u \in L^2(\mathcal{T}, X)$ with finite norm $\|u\|_{H^\alpha} = (\sum_{k=-\infty}^{\infty} (1 + k^2)^\alpha |\hat{u}(k)|^2)^{1/2}$. For integer values of α this is the same as $W^{\alpha, 2}(\mathcal{T}; X)$.
- $l^2(\mathbf{Z}; X)$ The Hilbert space of sequences $\{\hat{u}(k)\}_{k=-\infty}^{\infty}$ with finite norm induced by the inner product $\langle \hat{u}, \hat{v} \rangle_{l^2(\mathbf{Z}; X)} = \sum_{k=-\infty}^{\infty} \langle \hat{u}(k), \hat{v}(k) \rangle_X$.
- $\hat{u}$ The sequence of Fourier coefficients of $u \in L^2(\mathcal{T}; X)$. The k th Fourier coefficient $u(k)$ is defined as $\hat{u}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} u(t) dt$. Recall that the finite Fourier transform is an isomorphism between $L^2(\mathcal{T}; X)$ and $l^2(\mathbf{Z}; X)$.
- $M(\mathbf{R}; \mathbf{C})$ The set of (finite) complex measures on $\mathbf{R}$. The norm in $M(\mathbf{R}; \mathbf{C})$ is the total variation norm.
- $\hat{\mu}$ The Fourier transform of $\mu \in M(\mathbf{R}; \mathbf{C})$, $\hat{\mu}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} \mu(dt)$ for $\omega \in \mathbf{R}$.
- $\mathcal{D}(A)$ Domain of A . Here A is a closed, densely defined operator from a Hilbert space X into itself. We shall assume that A has a continuous inverse. Then one can use the norm $\|u\|_{\mathcal{D}(A)} = \|Au\|$ in $\mathcal{D}(A)$. This makes $\mathcal{D}(A)$ a Hilbert space continuously embedded in X .
- $\rho(A)$ Resolvent set of A ; the set of complex numbers λ for which $\lambda I - A$ has a continuous inverse. In particular, it is assumed that $0 \in \rho(A)$.
- $\sigma(A)$ Spectrum of A ; the complement of $\rho(A)$.
- $\mathcal{A}$ The operator $u \mapsto Au = \mu * u' + \nu * u + \eta * Au$, mapping $H^1(\mathcal{T}; X) \cap L^2(\mathcal{T}; \mathcal{D}(A))$ continuously into $L^2(\mathcal{T}; X)$. Here A is a closed, densely defined linear operator in a Hilbert space X , and $\mu, \nu, \eta \in M(\mathbf{R}; \mathbf{C})$.
- D The sequence of operators $D(k) = (ik\hat{\mu}(k) + \hat{\nu}(k))I + \hat{\eta}(k)A$, $k \in \mathbf{Z}$. As is well known, for each $u \in H^1(\mathcal{T}; X) \cap L^2(\mathcal{T}; \mathcal{D}(A))$ the k th Fourier coefficient of Au is $D(k)\hat{u}(k)$.

$$\begin{aligned}\mathcal{X} &= L^2(\mathcal{T}; X), \\ \mathcal{Y} &= H^1(\mathcal{T}; X) \cap L^2(\mathcal{T}; \mathcal{D}(A)).\end{aligned}$$

2. WELL-POSEDNESS

First, let us define what is meant by a strong solution of (1.1).

Definition 2.1. A function $u \in \mathcal{X} \equiv L^2(\mathcal{T}; X)$ is called a strong solution of (1.1) if $u \in \mathcal{Y} \equiv H^1(\mathcal{T}; X) \cap L^2(\mathcal{T}; \mathcal{D}(A))$ and (1.1) is satisfied.

Definition 2.2. The problem (1.1) is strongly well posed in $\mathcal{X}$ iff it is true that for every $f \in \mathcal{X}$ equation (1.1), i. e., the equation $Au = f$ has a unique strong solution.

By the closed graph theorem, if (1.1) is strongly well posed, then $u \in \mathcal{Y}$ depends continuously on $f \in \mathcal{X}$. In other words, $\mathcal{A}$ has a continuous inverse $\mathcal{B}$ that maps $\mathcal{X}$ into $\mathcal{Y}$. Another milder concept of well-posedness, one that corresponds to the concept that people usually think about when talking about well-posedness, is the following.

Definition 2.3. The problem (1.1) is (mildly) well posed in $\mathcal{X}$ iff there exists a linear operator $\mathcal{B}$ that maps $\mathcal{X}$ continuously into itself as well as $\mathcal{Y}$ into itself and that satisfies $\mathcal{B}\mathcal{A}u = \mathcal{A}\mathcal{B}u = u$ for all $u \in \mathcal{Y}$. In this case the function $\mathcal{B}f$ is called the mild solution of (1.1) and $\mathcal{B}$ is called the solution operator.

Clearly, the solution operator $\mathcal{B}$ above is unique, if it exists. For an integral representation of $\mathcal{B}$ in a particular finite-dimensional case, see Gripenberg, Londen, and Staffans [2, Theorem 3.11, pp. 87–88].

Observe that if (1.1) is well posed in $\mathcal{X}$, then the operator $\mathcal{A}$ can be extended to a closed operator in $\mathcal{X}$, whose inverse is the solution operator $\mathcal{B}$. Call this operator $\overline{\mathcal{A}}$. Clearly, if the problem is strongly well posed, then $\overline{\mathcal{A}} = \mathcal{A}$ and $\mathcal{D}(\overline{\mathcal{A}}) = \mathcal{D}(\mathcal{A}) = \mathcal{Y}$. In general $\mathcal{Y} \subset \mathcal{D}(\overline{\mathcal{A}}) \subset \mathcal{X}$.

Two necessary conditions for well-posedness are that the operator $\mathcal{A}$ is one-to-one and has a dense range. These conditions are easy to check.

Lemma 2.4. Let $\mathcal{A}$ be the continuous operator that maps $u \in \mathcal{Y}$ into $Au = \mu * u' + \nu * u + \eta * Au \in \mathcal{X}$.

- (i) $\mathcal{A}$ is one-to-one if and only if for each $k \in \mathbf{Z}$ the operator $D(k) = (ik\hat{\mu}(k) + \hat{\nu}(k))I + \hat{\eta}(k)A$ is one-to-one on $\mathcal{D}(A)$.
- (ii) The range of $\mathcal{A}$ is dense in $\mathcal{X}$ if and only if for each $k \in \mathbf{Z}$ the range of the operator $D(k)$ mapping $\mathcal{D}(A)$ into X is dense in X .

The easy proof is left to the reader (cf. Burton, Krisztin, and Zhang [1, Theorem 2.2]).

3. THE MAIN RESULTS

The following three theorems are the main results of this paper.

Theorem 3.1. Problem (1.1) is well posed in $\mathcal{X}$ if and only if for each $k \in \mathbf{Z}$ the operator $D(k) = (ik\hat{\mu}(k) + \hat{\nu}(k))I + \hat{\eta}(k)A$ has a continuous inverse and

$$(3.1) \quad \lambda \stackrel{\text{def}}{=} \sup_{k \in \mathbf{Z}} \|D(k)^{-1}\| < \infty.$$

When this is the case, the norm of the solution operator $\mathcal{B}$ is λ .

Observe that this theorem implies that if (1.1) is well posed (and $\hat{\eta}(k) \neq 0$ for some k), then A has a nonempty resolvent set. By adding a multiple of the identity to A and subtracting the corresponding multiple of η from ν , we may assume that $0 \in \rho(A)$, i.e., A has a continuous inverse. Thus, we lose no generality by assuming (as we did in the definition of $\mathcal{D}(A)$ in §1) that A has a continuous inverse. (The proof of Theorem 3.1 does not use this assumption.)

Theorem 3.2. *Problem (1.1) is strongly well posed in $\mathcal{X}$ if and only if it is well posed and, in addition,*

- (i) $\sup_{k \in \mathbb{Z}} |k| \|D(k)^{-1}\| < \infty$,
- (ii) $\hat{\eta}(k) \neq 0$ for all $k \in \mathbb{Z}$, or the operator A is bounded, and
- (iii) $\sup_{k \in \mathbb{Z}} \|AD(k)^{-1}\| < \infty$.

Theorem 3.3. *Suppose that (1.1) is well posed in $\mathcal{X}$. Then the solution operator $\mathcal{B}$ is compact from $\mathcal{X}$ into itself if and only if the unit ball of $\mathcal{D}(A)$ is compact in X and, in addition,*

- (i) $\hat{\eta}(k) \neq 0$ for all $k \in \mathbb{Z}$, or X is finite dimensional, and
- (ii) $\|D(k)^{-1}\| \rightarrow 0$ as $|k| \rightarrow \infty$.

Of course, if X is finite dimensional then the assumption that the unit ball of $\mathcal{D}(A)$ is compact in X is redundant.

4. PROOFS

Proof of Theorem 3.1. Suppose that (1.1) is well posed. Let $f(t) = e^{ikt}y$ for some $k \in \mathbb{Z}$ and $y \in \mathcal{D}(A)$, and let $u = \mathcal{B}f$. Then $u \in \mathcal{Y}$ and the Fourier coefficients of u satisfy

$$D(j)\hat{u}(j) = \begin{cases} y & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 2.4, this implies that $\hat{u}(j) = 0$ for $j \neq k$, i.e., u is of the form $u(t) = e^{ikt}x$ for some $x \in \mathcal{D}(A)$. Note that $D(k)x = y$. By the well-posedness assumption, $x \in X$ depends continuously on $y \in X$, since $\mathcal{B}$ maps $\mathcal{X}$ continuously into itself. This being true for a dense set of vectors y (i.e., for all $y \in \mathcal{D}(A)$), the operator $D(k)$ has a continuous inverse in X . Moreover, since

$$\|D(k)^{-1}y\|_X = \|x\|_X = \|\mathcal{B}f\|_{\mathcal{X}} \leq \|\mathcal{B}\| \|f\|_{\mathcal{X}} = \|\mathcal{B}\| \|y\|_X,$$

we have $\|D(k)^{-1}\| \leq \|\mathcal{B}\|$. This proves (3.1), and half of the final statement of Theorem 3.1.

Conversely, suppose that (3.1) is true. For each $f \in \mathcal{X}$, let $\mathcal{B}$ be the operator that maps f into the function $u \in \mathcal{X}$ whose k th Fourier coefficient is $D(k)^{-1}\hat{f}(k)$. Then $\mathcal{B}$ has the desired properties. $\square$

Proof of Theorem 3.2. The easy proof of this theorem is similar to the proof given above, and it is left to the reader. Observe that (i) holds if and only if $\mathcal{B}$ maps $\mathcal{X}$ continuously into $H^1(\mathcal{T}; X)$ and (ii) and (iii) hold if and only if $\mathcal{B}$ maps $\mathcal{X}$ continuously into $L^2(\mathcal{T}; \mathcal{D}(A))$. $\square$

The same proof can be applied to get a slightly more general result.

Theorem 4.1. *Let (1.1) be well posed in $\mathcal{X}$, and let $\alpha \geq 0$. Then the following two claims are true:*

(i) *The solution operator $\mathcal{B}$ maps $\mathcal{X}$ continuously into $H^\alpha(\mathcal{T}; X)$ if and only if*

$$\lambda \stackrel{\text{def}}{=} \sup_{k \in \mathbb{Z}} (1 + k^2)^{\alpha/2} \|D(k)^{-1}\| < \infty.$$

The norm of the solution operator $\mathcal{B}$, regarded as an operator from $\mathcal{X}$ into $H^\alpha(\mathcal{T}; X)$, is λ .

(ii) *The solution operator $\mathcal{B}$ maps $\mathcal{X}$ continuously into $H^\alpha(\mathcal{T}; \mathcal{D}(A))$ if and only if Theorem 3.2(ii) holds and, in addition,*

$$\lambda \stackrel{\text{def}}{=} \sup_{k \in \mathbb{Z}} (1 + k^2)^{\alpha/2} \|AD(k)^{-1}\| < \infty.$$

The norm of $\mathcal{B}$, regarded as an operator from $\mathcal{X}$ into $H^\alpha(\mathcal{T}; \mathcal{D}(A))$, is λ .

The proof of this theorem, too, is left to the reader.

Proof of Theorem 3.3. Suppose that the unit ball of $\mathcal{D}(A)$ is compact in X and that (i) and (ii) hold. Then, for each K , the operator $\mathcal{B}_K$ that maps f into

$$(\mathcal{B}_K f)(t) = \sum_{k=-K}^K D(k)^{-1} \hat{f}(k) e^{ikt}$$

is a finite sum of compact operators, hence compact. Because of (ii), as $K \rightarrow \infty$, $\mathcal{B}_K$ converges in norm to $\mathcal{B}$, so $\mathcal{B}$ is compact.

Conversely, suppose that the solution operator $\mathcal{B}$ is compact. Let P_k be the continuous projection that maps $f \in \mathcal{X}$ into $(P_k f)(t) = \hat{f}(k) e^{ikt}$. The compactness of $\mathcal{B}$ implies that $\mathcal{B}P_k$ is compact. But

$$(\mathcal{B}P_k f)(t) = D(k)^{-1} \hat{f}(k) e^{ikt};$$

hence $D(k)^{-1}$ is compact. When X is infinite dimensional, this implies that $\hat{f}(k) \neq 0$ and that the unit ball of $\mathcal{D}(A)$ is compact in X .

Suppose, in addition, that (ii) is not true. Then there is some $\varepsilon > 0$, some sequence $k_j \rightarrow \pm\infty$, and some sequence $y_j \in X$, with $\|y_j\| = 1$, such that $\|D(k_j)^{-1} y_j\| \geq \varepsilon$. Consider the sequence $f_j(t) = y_j e^{ik_j t}$. This is a bounded sequence in $\mathcal{X}$, hence by the compactness of $\mathcal{B}$, $\mathcal{B}f_j$ contains a convergent subsequence. To simplify the notations, let us replace the original sequence by the new one, so that the sequence $\mathcal{B}f_j$ itself converges. Recall that $(\mathcal{B}f_j)(t) = D(k_j)^{-1} y_j e^{ik_j t}$, hence

$$\|\mathcal{B}f_j\|_{\mathcal{X}} = \|D(k_j)^{-1} y_j\|_X \geq \varepsilon.$$

Thus $\mathcal{B}f_j$ cannot tend to zero. However, since every Fourier coefficient of $\mathcal{B}f_j$ is zero apart from the k_j th one, and since $k_j \rightarrow \pm\infty$, every Fourier coefficient of the limit must be zero, hence the limit must be zero. This contradiction proves that (ii) cannot be false when $\mathcal{B}$ is compact. $\square$

One can use a similar proof to show that a subset $U \subset \mathcal{X}$ is totally bounded if and only if

- (i) for each $k \in \mathbb{Z}$, the set $\{\hat{u}(k) | u \in U\}$ is totally bounded in X , and
- (ii) $\sup_{u \in U} \sum_{|k| \geq K} \|\hat{u}(k)\|^2 \rightarrow 0$ as $K \rightarrow \infty$.

5. THE CASE WHERE $-A$ GENERATES AN ANALYTIC SEMIGROUP

There are several possible extension of the results proved above, the proofs remaining essentially the same. We shall discuss below one such case, the case where $-A$ generates an analytic semigroup (replacing η by $-\eta$, one may assume instead that A itself generates an analytic semigroup).

Whenever $-A$ generates an analytic semigroup, there are constants $0 < \delta < \pi/2$ and $M > 0$ such that (see, e.g., Pazy [3, Theorem 5.2, pp. 61–62]; if necessary, add a multiple of the identity to A)

$$\rho(-A) \supset \Sigma = \left\{ \lambda \mid |\arg \lambda| < \frac{\pi}{2} + \delta \right\} \cup \{0\}$$

and

$$(5.1) \quad \|[\lambda I + A]^{-1}\| \leq \frac{M}{1 + |\lambda|} \quad \text{for } \lambda \in \Sigma.$$

For $\beta \geq 0$, define the fractional powers A^β as in Pazy [3, §2.6].

The easy proof of the following theorem is left to the reader (cf. the proofs given in the preceding section).

Theorem 5.1. *Let $-A$ generate an analytic semigroup, let α and β be nonnegative constants, and let the problem (1.1) be well posed in $\mathcal{X}$. Then the solution operator $\mathcal{B}$ maps $\mathcal{X}$ into $H^\alpha(\mathcal{T}; \mathcal{D}(A^\beta))$ if and only if*

- (i) $\hat{\eta}(k) \neq 0$ for all $k \in \mathbb{Z}$, or the operator A^β is bounded (i.e., $\beta = 0$, or A is bounded), and
- (ii) $\lambda \stackrel{\text{def}}{=} \sup_{k \in \mathbb{Z}} (1 + k^2)^{\alpha/2} \|A^\beta D(k)^{-1}\| < \infty$.

The norm of $\mathcal{B}$, regarded as an operator from $\mathcal{X}$ into $H^\alpha(\mathcal{T}; \mathcal{D}(A^\beta))$, is λ .

One particular interesting case of this theorem is the case where $\alpha + \beta = 1$, because this case is related to the strong well-posedness of (1.1).

Theorem 5.2. *Let $-A$ generate an analytic semigroup and let the problem (1.1) be strongly well posed in $\mathcal{X}$. Then for all $\alpha \in [0, 1]$ the solution operator $\mathcal{B}$ maps $\mathcal{X}$ continuously into $H^\alpha(\mathcal{T}; \mathcal{D}(A^{1-\alpha}))$.*

Proof. Since the problem is strongly well posed, the operator that maps u into $(\mathcal{B}u)' + A\mathcal{B}u$ is bounded on $\mathcal{X}$. Thus, to prove the claim it suffices to prove that the inverse of the operator $u \mapsto u' + Au$ is continuous from $\mathcal{X}$ into $H^\alpha(\mathcal{T}; \mathcal{D}(A^{1-\alpha}))$. In other words, it suffices to prove the theorem in the special case where equation (1.1) is of the form $u' + Au = f$. This is (1.1) with $\mu = \eta = \delta$ (the unit point mass at zero) and $\nu = 0$.

By Pazy [3, Theorem 6.10, p. 73], there is some constant $C \geq 1$ such that for every x in X and every $\alpha \in [0, 1]$,

$$\begin{aligned} \|A^{1-\alpha}(ikI + A)^{-1}x\| &\leq C\|(ikI + A)^{-1}x\|^\alpha \|A(ikI + A)^{-1}x\|^{1-\alpha} \\ &\leq C\|(ikI + A)^{-1}\|^\alpha \|A((ikI + A)^{-1})^{1-\alpha}\|x\|. \end{aligned}$$

Because of (5.1), and since

$$\sup_{k \in \mathbb{Z}} \|A(ikI + A)^{-1}\| < \infty,$$

it is possible to apply Theorem 5.1 with $\beta = 1 - \alpha$. $\square$

6. APPLICATION TO A NONLINEAR PROBLEM

By applying the preceding results one easily proves the following

Theorem 6.1. *Let (1.1) be strongly well posed in $\mathcal{X}$, and let the unit ball of $\mathcal{D}(A)$ be compact in X . Let f be a continuous mapping of $\mathcal{X}$ into itself, and suppose that for some $M > 0$,*

$$\sup_{\|u\| \leq M} \|f(u)\| \leq M/\|\mathcal{B}\|.$$

Then the equation

$$\mu * u' + \nu * u + \eta * Au = f(u)$$

has a strong solution (i.e., one that belongs to $\mathcal{Y}$), with $\|u\|_{\mathcal{X}} \leq M$.

Proof. Combining Theorems 2.6 and 2.7 one finds that the solution operator $\mathcal{B}$ of (1.1) is compact. Thus, one may apply Schauder's fixed point theorem to the equation $u = \mathcal{B}f(u)$ in the ball $\{u \in \mathcal{X} \mid \|u\| \leq M\}$ to get the existence of a strong solution. $\square$

This theorem can be used to both simplify and improve Burton, Krisztin, and Zhang [1, Theorems 2.1 and 4.1].

Further extensions are possible. For example, when $-A$ generates an analytic semigroup one may let f be an unbounded operator on $\mathcal{X}$. It suffices if, e.g., f is continuous from $H^\alpha(\mathcal{T}; \mathcal{D}(A^\beta))$ into $\mathcal{X}$, where α and β are nonnegative constants satisfying $\alpha + \beta < 1$, and f maps some ball in $H^\alpha(\mathcal{T}; \mathcal{D}(A^\beta))$ into a sufficiently small ball of $\mathcal{X}$. To see this, use Theorem 5.2 and the fact that $H^\alpha(\mathcal{T}; \mathcal{D}(A^\beta))$ is compact in $H^{\alpha'}(\mathcal{T}; \mathcal{D}(A^{\beta'}))$ whenever the unit ball of $\mathcal{D}(A)$ is compact in X and $\alpha > \alpha' \geq 0$ and $\beta > \beta' \geq 0$. (The proof of this compactness claim is very similar to the proof of Theorem 3.3.)

REFERENCES

1. T. A. Burton, T. Krisztin, and B. Zhang, *Periodic solutions of an integrodifferential equation with application to heat flow* (to appear).
2. G. Gripenberg, S.-O. Londen, and O. J. Staffans, *Volterra integral and functional equations*, Cambridge Univ. Press, Cambridge, 1990.
3. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York and Berlin, 1983.
4. O. J. Staffans, *Some well-posed functional equations that generate semigroups*, J. Differential Equations **58** (1985), 157–191.
5. —, *Periodic solutions of an abstract integrodifferential equation*, Functional Differential Equations and Related Topics (Proc. Internat. Conf., Kyoto 1990), World Scientific, Singapore, 1991, pp. 344–348.

INSTITUTE OF MATHEMATICS, HELSINKI UNIVERSITY OF TECHNOLOGY, SF-02150 ESPOO 15, FINLAND

E-mail address: olof.staffans@hut.fi

DEPARTMENT OF MATHEMATICS, VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY, BLACKSBURG, VIRGINIA 24061–0123