AN EXAMPLE OF A HARMONIC MAP BETWEEN EUCLIDEAN BALLS

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ABSTRACT. Here we construct a homeomorphism of the unit sphere in three-dimensional Euclidean space to itself whose harmonic extension in the unit ball is not a diffeomorphism.

A theorem of Rado [R] states that if $\Omega \subseteq \mathbb{R}^2$ is a bounded convex domain with smooth boundary $\partial\Omega$ and if $\varphi: S^1 \rightarrow \partial\Omega$ is a homeomorphism then its harmonic extension $u: D \rightarrow \Omega$, where D is the unit disc, is a diffeomorphism.

Here we will show that this is false in the 3-dimensional case. In fact, we will construct a homeomorphism $\varphi: S^2 \rightarrow S^2$ such that its harmonic extension $u: B^3 \rightarrow B^3$, where B^3 is the open unit 3-ball, is not a diffeomorphism.

Of course, this leaves open the question of whether the harmonic extension of a homeomorphism $\varphi: S^2 \rightarrow S^2$ is a homeomorphism.

Notation. S^2 denotes the 2-sphere of radius 1 in $\mathbb{R}^3$ and B^3 the open unit ball in $\mathbb{R}^3$. For a continuous function $f: S^2 \rightarrow \mathbb{R}$ its harmonic extension is the unique continuous function $v: \overline{B^3} \rightarrow \mathbb{R}$ such that $\Delta v = 0$ in B^3 and $v = f$ on $\partial B^3 = S^2$. For a C^1 function $g: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, J_g denotes its Jacobian matrix.

Theorem. *There exists a homeomorphism $\varphi: S^2 \rightarrow S^2$ whose harmonic extension $u: B^3 \rightarrow B^3$ satisfies $\det J_u(0) = 0$, hence it is not a diffeomorphism.*

Proof. Given a continuous function $f: S^2 \rightarrow \mathbb{R}$, its harmonic extension $v: B^3 \rightarrow \mathbb{R}$ is given by

$$(1) \quad v(x) = \frac{1 - |x|^2}{4\pi} \int_{S^2} \frac{f(y)}{|x - y|^3} d\sigma(y)$$

for $x \in B^3$, where $d\sigma$ denotes the surface measure on S^2 . Hence

$$(2) \quad \nabla v(0) = \frac{3}{4\pi} \int_{S^2} x f(x) d\sigma(x).$$

Now choose coordinate axes x_1, x_2, x_3 so that 0 is the center of B^3 and let r, θ denote polar coordinates on the x_1x_2 -plane. Let $\delta > 0$ be small enough so that if

$$(3) \quad E = \{x \in S^2 : x_3 \in [-1 + \delta, 0] \cup [1 - \delta, 1]\}$$

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then

$$(4) \quad \int_E x_3 d\sigma(x) < 0.$$

By (4) there exists $\eta > 0$ sufficiently small so that if

$$(5) \quad \begin{aligned} F = & \{x \in S^2 : x_3 \in [-1 + \delta, 0], |\theta| \geq \eta, |\pi - \theta| \geq \eta\} \\ & \cup \left\{x \in S^2 : x_3 \in [0, 1 - \delta], \left|\theta - \frac{\pi}{2}\right| \leq \eta, \left|\theta + \frac{\pi}{2}\right| \leq \eta\right\} \\ & \cup \{x \in S^2 : x_3 \in [1 - \delta, 1]\} \end{aligned}$$

then

$$(6) \quad \int_F x_3 d\sigma(x) < 0.$$

The interior of F and $S^2 \setminus F$ are simply connected domains in S^2 whose common boundary is a Jordan curve $\gamma: S^1 \rightarrow S^2$. Moreover, γ is symmetric with respect to the x_1x_3 - and x_2x_3 -planes. Also $(0, 0, 1) \in \text{int } F$ and $(0, 0, -1) \in S^2 \setminus F$. Therefore there exist homotopies

$$C_t: S^1 \rightarrow F \quad \text{and} \quad C_{-t}: S^1 \rightarrow \overline{S^2 \setminus F}, \quad \text{where } 0 \leq t \leq 1,$$

having the following properties:

- (i) $C_0 = \gamma$;
- (ii) C_t, C_{-t} are Jordan curves and are symmetric with respect to the x_1x_3 - and x_2x_3 -planes for $0 \leq t < 1$;
- (iii) $C_1(S^1) = \{(0, 0, 1)\}$ and $C_{-1}(S^1) = \{(0, 0, -1)\}$;
- (iv) $C_{t_1}(S^1) \cap C_{t_2}(S^1) = \emptyset$ if $-1 \leq t_1 < t_2 \leq 1$;
- (v) $S^2 = \bigcup_{-1 \leq t \leq 1} C_t$.

Then for any continuous strictly increasing $h: [-1, 1] \rightarrow [-1, 1]$ such that $h(-1) = -1$ and $h(1) = 1$, let $\varphi_h: S^2 \rightarrow [-1, 1]$ be defined by

$$(7) \quad \varphi_h(x) = h(t) \quad \text{if } x \in C_t, \quad -1 \leq t \leq 1.$$

By (ii), (iii), (iv), and (v), φ_h is well defined and continuous on S^2 and, moreover, it is even in the x_1 and x_2 variables. Hence

$$(8) \quad \int_{S^2} x_i \varphi_h(x) d\sigma(x) = 0 \quad \text{for } i = 1, 2.$$

By (iii) there exists $0 < \varepsilon_1 < 1$ such that $C_t \subseteq \{x \in S^2 : x_3 > \frac{1}{2}\}$ and $C_{-t} \subseteq \{x \in S^2 : x_3 < -\frac{1}{2}\}$ if $1 - \varepsilon_1 \leq t \leq 1$. Hence there exist $\varepsilon_2 > 0$ sufficiently small such that if $h_0: [-1, 1] \rightarrow [-1, 1]$ is defined by $h_0(-1) = -1$, $h_0(-1 + \varepsilon_1) = -\varepsilon_2$, $h_0(1 - \varepsilon_1) = \varepsilon_2$, $h_0(1) = 1$, and h_0 is linear in the intervals determined by the points $-1, -1 + \varepsilon_1, 1 - \varepsilon_1, 1$, then

$$(9) \quad \int_{S^2} x_3 \varphi_{h_0}(x) d\sigma(x) > 0.$$

By (6) there exist $0 < \varepsilon_3 < 1$ and $\varepsilon_4 > 0$ sufficiently small such that if $h_1: [-1, 1] \rightarrow [-1, 1]$ is defined by $h_1(-1) = -1$, $h_1(-1 + \varepsilon_3) = -\varepsilon_4$, $h_1(0) = 0$, $h_1(\varepsilon_3) = 1 - \varepsilon_4$, $h_1(1) = 1$, and h_1 is linear in the intervals determined by the points $-1, -1 + \varepsilon_3, 0, \varepsilon_3, 1$, then

$$(10) \quad \int_{S^2} x_3 \varphi_{h_1}(x) d\sigma(x) < 0.$$

By (9), (10) there exists $s \in (0, 1)$ such that if $h_s = sh_0 + (1-s)h_1$ then

$$(11) \quad \int_{S^2} x_3 \varphi_{h_s}(x) d\sigma(x) = 0.$$

Then $h_s: [-1, 1] \rightarrow [-1, 1]$ is continuous strictly increasing, $h_s(-1) = -1$, and $h_s(1) = 1$. Hence by the properties of the C_t 's, $-1 \leq t \leq 1$, it follows that there exists a homeomorphism $\varphi: S^2 \rightarrow S^2$ such that if $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ then $\varphi_3 = \varphi_{h_s}$. By (8) and (11), it follows that

$$(12) \quad \int_{S^2} x \varphi_3(x) d\sigma(x) = 0.$$

Let $u: B^3 \rightarrow B^3$ be the harmonic extension of φ . Then by (2) and (12) it follows that

$$(13) \quad \nabla u_3(0) = 0$$

and hence

$$(14) \quad \det J_u(0) = 0.$$

Hence u is not a diffeomorphism.

Remarks. 1. This example can be easily generalized to the n -dimensional case for $n \geq 3$. Also by being more careful in the construction of the C_t 's and h_0, h_1 , it is possible to construct a smooth diffeomorphism $\varphi: S^2 \rightarrow S^2$ whose harmonic extension is not a diffeomorphism.

2. Actually we have constructed a homeomorphism $\varphi: S^2 \rightarrow S^2$ whose third coordinate $\varphi_3: S^2 \rightarrow [-1, 1]$ is orthogonal to the coordinate functions x_1, x_2, x_3 .

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