THE IDEAL DETERMINED BY THE UNSYMMETRIC GAME

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(Communicated by Andreas R. Blass)

ABSTRACT. In the present paper we study the ideal of all subsets of $\mathcal{X}^\omega$ for which the second player has a winning strategy in the unsymmetric game. We describe its cardinal coefficients and the notions of forcing determined by it.

0. INTRODUCTION

Let $\Gamma_{\mathcal{X}}^*(A)$ denote the unsymmetric game on the set $A \subseteq \mathcal{X}^\omega$. In this game Player I chooses finite sequences of elements of $\mathcal{X}$ while Player II chooses single elements of $\mathcal{X}$. They play alternately to construct a function $c = s_1 \wedge n_1 \wedge s_2 \wedge n_2 \wedge \dots \in \mathcal{X}^\omega$, where s_i is the i th sequence chosen by the first player and n_i is the answer of the second. Player I wins if and only if the result c of the game belongs to A .

One can easily observe that the family of all $A \subseteq \mathcal{X}^\omega$ for which Player II has a winning strategy in $\Gamma_{\mathcal{X}}^*(A)$ is a σ -ideal. Denote this ideal by $\mathfrak{D}_{\mathcal{X}}$. Davis proved that if the second player had a winning strategy in $\Gamma_{\mathcal{X}}^*(A)$ then he had one that did not depend on the history of the game, a strategy that did not need to know who played which element (cf. [D]). Thus

$$\mathfrak{D}_{\mathcal{X}} = \{A \subseteq \mathcal{X}^\omega : (\exists f : \mathcal{X}^{<\omega} \rightarrow \mathcal{X})(\forall c \in A) \\ (\exists n \in \omega)(\forall m > n)(c(m) \neq f(c|_m))\}.$$

The ideal $\mathfrak{D}_{\mathcal{X}}$ appeared implicitly in Mycielski's proof of the determinacy of analytic sets in games $\Gamma_{\mathcal{X}}^*$ (cf. [M]); it was studied in [R1]. In the present paper we continue those investigations.

Our notation and terminology are standard and derived from [J].

Let $\mathcal{X}$ be a countable set with at least two elements. We will think of $\mathcal{X}$ as an element of $\omega \cup \{\omega\}$.

1. GENERAL PROPERTIES

The space $\mathcal{X}^\omega$ is endowed with the product topology determined by the basis $\{[s] : s \in \mathcal{X}^{<\omega}\}$, where $[s] = \{c \in \mathcal{X}^\omega : s \subseteq c\}$. If $\mathcal{X}$ is finite then the Lebesgue measure λ on the space $\mathcal{X}^\omega$ is the obvious product measure. In the space ω^ω , λ is the product of the measure λ_0 on ω defined by $\lambda_0(\{n\}) = 2^{-(n+1)}$. Let $\mathbb{K}(\mathcal{X}^\omega)$ and $\mathbb{L}(\mathcal{X}^\omega)$ stand for σ -ideals of first category (meager) sets of the

Received by the editors December 26, 1990 and, in revised form, July 2, 1991.

1991 *Mathematics Subject Classification*. Primary 04A15; Secondary 03E40, 90D13.

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space $\mathcal{X}^\omega$ and of Lebesgue null sets in the space $\mathcal{X}^\omega$. $\text{BOREL}(\mathcal{X}^\omega)$ is the family of all Borel subsets of $\mathcal{X}^\omega$. If $\mathbb{I}$ is a σ -ideal on $\mathcal{X}^\omega$ then $\text{BOREL}(\mathbb{I})$ denotes the σ -algebra generated by $\text{BOREL} \cup \mathbb{I}$. Two ideals $\mathbb{I}$ and $\mathbb{J}$ on $\mathcal{X}^\omega$ are *orthogonal* if there exists sets $A \in \mathbb{I}$ and $B \in \mathbb{J}$ such that $A \cup B = \mathcal{X}^\omega$.

Recall some results concerning $\mathcal{D}_{\mathcal{X}}$ from [R1].

Theorem 1.1. (a) $\mathcal{D}_{\mathcal{X}}$ is a σ -ideal with a Borel basis. If $\mathcal{X}$ is a group then $\mathcal{D}_{\mathcal{X}}$ is translation invariant and symmetric.

(b) $\mathcal{D}_{\mathcal{X}} \subseteq \mathbb{K}(\mathcal{X}^\omega)$. If $\mathcal{X}$ is finite then $\mathcal{D}_{\mathcal{X}} \subseteq \mathbb{L}(\mathcal{X}^\omega)$.

(c) $\mathcal{D}_\omega$ is orthogonal to $\mathbb{L}(\omega^\omega)$ and contains all compact subsets of ω^ω . $\mathcal{D}_2 = [2^\omega]^{\leq \omega}$.

(d) There exist c disjoint closed subsets of $\mathcal{X}^\omega$ that do not belong to $\mathcal{D}_{\mathcal{X}}$.

(e) $\text{BOREL}(\mathcal{D}_{\mathcal{X}})$ is not closed under the Souslin operation $\mathcal{A}$.

A *tree* on $\mathcal{X}$ is a subset T of $\mathcal{X}^{<\omega}$ such that $(\forall s \in T)(\forall t \subseteq s)(t \in T)$. If s is an element of a tree T then $\text{succ}_T(s) = \{x \in \mathcal{X} : s^\wedge x \in T\}$. The *body* $[T]$ of a tree T is a set of all infinite branches through T , $[T] = \{c \in \mathcal{X}^\omega : (\forall n \in \omega)(c|n \in T)\}$.

It follows that each set from $\mathcal{D}_{\mathcal{X}}$ can be covered by the countable union of bodies of trees on $\mathcal{X}$ which satisfy

$$(*) \quad (\forall s \in T)(\text{succ}_T(s) \neq \mathcal{X}).$$

Clearly, $[T] \in \mathcal{D}_{\mathcal{X}}$ provided T satisfies $(*)$. On the other hand, if the first player has a winning strategy in $\Gamma_{\mathcal{X}}^*(A)$ then A contains the body of a tree T on $\mathcal{X}$ such that

$$(**) \quad (\forall s \in T)(\exists t \in T)(s \subseteq t \ \& \ \text{succ}_T(t) = \mathcal{X}).$$

Obviously, Player I can win $\Gamma_{\mathcal{X}}^*([T])$ provided T satisfies $(**)$.

Theorem 1.2 [M]. If $A \in \Sigma_1^1(\mathcal{X}^\omega)$ then either $A \in \mathcal{D}_{\mathcal{X}}$ or there exists a tree T on $\mathcal{X}$ such that $[T] \subseteq A$ and T satisfies $(**)$.

Proposition 1.3. (a) Suppose $\mathcal{X} \subseteq \mathcal{Y}$. Then there exists a continuous function $\phi: \mathcal{Y}^\omega \rightarrow \mathcal{X}^\omega$ such that $\phi^{-1}[A] \in \mathcal{D}_{\mathcal{Y}}$ provided $A \in \mathcal{D}_{\mathcal{X}}$.

(b) There exists a continuous function $\psi: \omega^\omega \rightarrow \omega^\omega$ such that $\psi^{-1}[A] \in \mathcal{D}_\omega$ provided $A \in \mathbb{K}(\omega^\omega)$.

Proof. (a) Fix $x_0 \in \mathcal{X}$. Let $\phi': \mathcal{Y}^{<\omega} \rightarrow \mathcal{X}^{<\omega}$ be defined by

$$\phi'(s)(n) = \begin{cases} s(n), & s(n) \in \mathcal{X}, \\ x_0, & s(n) \notin \mathcal{X}, \end{cases}$$

for $s \in \mathcal{Y}^{<\omega}$ and $n < lh(s)$. For $c \in \mathcal{Y}^\omega$ put $\phi(c) = \bigcup_{n \in \omega} \phi'(c|n)$. Clearly, $\phi: \mathcal{Y}^\omega \rightarrow \mathcal{X}^\omega$ is continuous. Suppose that $A \in \mathcal{D}_{\mathcal{X}}$ and $f: \mathcal{X}^{<\omega} \rightarrow \mathcal{X}$ is a witness for it. Let $g = f \circ \phi': \mathcal{Y}^{<\omega} \rightarrow \mathcal{X}$. Then $g(c|n) = c(n)$ implies $f(\phi(c)|n) = \phi(c)(n)$ for $c \in \mathcal{Y}^\omega$, and hence g is a winning strategy for the second player in $\Gamma_{\mathcal{Y}}^*(\phi^{-1}[A])$.

(b) Let $\pi: \omega \rightarrow \omega^{<\omega}$ be a canonical bijection. Put $\psi'(s) = \pi(s(0))^\wedge \pi(s(1))^\wedge \dots^\wedge \pi(s(n))$ for $s \in \omega^{n+1}$ and $\psi(c) = \bigcup_{n > 0} \psi'(c|n)$ for $c \in \omega^\omega$. Obviously, $\psi: \omega^\omega \rightarrow \omega^\omega$ is continuous. Suppose that $A \in \mathbb{K}(\omega^\omega)$ and let $G = \bigcap_{n \in \omega} G_n$ be disjoint from A , where each G_n is open dense in ω^ω . For $s \in \omega^{<\omega}$ let

$f(s) \in \omega$ be such that $[\psi'(s) \wedge \pi(f(s))] \subseteq \bigcap_{n < l_{hs}} G_n$. Then $\psi(c) \in G$ provided

$$(\forall m)(\exists n > m)(f(c|n) = c(n)),$$

and hence f is a winning strategy for Player II in $\Gamma_\omega^*(\psi^{-1}[A])$. $\square$

2. THE NOTION OF FORCING DETERMINED BY $\mathbb{D}_{\mathcal{X}}$

Let us consider the factor Boolean algebra $\text{BOREL}(\mathcal{X}^\omega)/\mathbb{D}_{\mathcal{X}}$ as a notion of forcing.

Let $\mathbb{D}_{\mathcal{X}}$ be the family of all trees $T \subseteq \mathcal{X}^{<\omega}$ satisfying $(**)$ and such that for every $s \in T$ either $\text{succ}_T(s) = \mathcal{X}$ or $|\text{succ}_T(s)| = 1$. It is ordered by inclusion. By Theorem 1.2 the order $\mathbb{D}_{\mathcal{X}}$ can be densely embedded in $\text{BOREL}(\mathcal{X}^\omega)/\mathbb{D}_{\mathcal{X}}$; hence they have the same forcing properties.

Note that $\mathbb{D}_2$ is the Sacks perfect set forcing.

We define orders $\leq_n$ on $\mathbb{D}_{\mathcal{X}}$ considering two cases: $\mathcal{X}$ is finite and $\mathcal{X}$ is infinite.

Suppose first that $\mathcal{X} = k \in \omega$. For a tree $T \in \mathbb{D}_k$ let T^n contain all elements s of T such that $\text{succ}_T(s) = k$ and there exist exactly $n-1$ initial segments of s with the same property. We write $T_1 \leq_n T_2$ if and only if $T_1 \leq T_2$ and $(T_1)^n = (T_2)^n$.

Now consider the case $\mathcal{X} = \omega$. Fix an enumeration $\{s_n : n \in \omega\} = \omega^{<\omega}$ such that $s_i \subseteq s_j$ implies $i \leq j$ and such that $s_i = s^\wedge n$, $s_j = s^\wedge m$, $n < m$, imply $i < j$. Let $T \in \mathbb{D}_\omega$. Each element of $\omega^{<\omega}$ determines in a canonical way an element of $\{s \in T : \text{succ}_T(s) = \omega\}$; denote by $T(i)$ the element determined by s_i . For $T, T' \in \mathbb{D}_{\mathcal{X}}$ and $n \in \omega$ put $T \leq_n T'$ if and only if $T \leq T'$ and $\{T(i) : i \leq n\} = \{T'(i) : i \leq n\}$.

Clearly, orders $\leq_n$ have the fusion property and witness Baumgartner's Axiom A for $\mathbb{D}_{\mathcal{X}}$ (cf. [B]).

Let $\mathbb{ID}_{\mathcal{X}} = \{A \subseteq \omega^\omega : (\forall T \in \mathbb{D}_{\mathcal{X}})(\exists T' \in \mathbb{D}_{\mathcal{X}}, T' \leq T)([T'] \cap A = \emptyset)\}$. Clearly it is a σ -ideal containing $\mathbb{D}_{\mathcal{X}}$. Note that the formula " $c \in \omega^\omega$ is a code for a Borel set from $\mathbb{D}_{\mathcal{X}}$ " is absolute.

Theorem 2.1. (a) $\mathbb{D}_\omega$ adds a Cohen real.

(b) $\mathbb{D}_\omega \Vdash "2^\omega \cap V \notin \mathbb{K}."$ Consequently, $\mathbb{D}_\omega$ adds neither random reals nor dominating reals.

Proof. (a) Let $G \subseteq \mathbb{D}_\omega$ be a generic set over V , $c_G = \bigcup \{\text{root } T : T \in G\}$. Obviously $c_G : \omega \rightarrow \omega$. Since $\mathbb{D}_\omega \subseteq \mathbb{ID}_\omega$, $c_G \notin A$ provided $A \in \mathbb{D}_\omega$ is a Borel set coded in V . Let ψ be a function given by Proposition 1.3(b). We claim that $\psi(c_G)$ is a Cohen real over V . Indeed, if $A \in \mathbb{K}(2^\omega)$ is a Borel set coded in V , then $\psi^{-1}[A] \in \mathbb{D}_\omega$ is a Borel set coded in V , too. Hence $c_G \notin \psi^{-1}[A]$.

(b) Assume that $T \Vdash "\dot{G}_n$ are decreasing open dense subsets of 2^ω for $n \in \omega$." Construct inductively sequences $\langle T_n : n \in \omega \rangle$, $\langle S_n : n \in \omega \rangle \subseteq \mathbb{D}_\omega$, $\langle v_n : n \in \omega \rangle \subseteq 2^{<\omega}$, and $\langle u_s : s \in \omega^{<\omega} \rangle \subseteq \omega^{<\omega}$ such that

(0) $u_{s_0} = \text{root } T$, $S_0 \leq T|u_{s_0}$, and $v_0 \in 2^{<\omega}$ satisfy $S_0 \Vdash "[v_0] \subseteq \dot{G}_0,"$ T_0 is a modification of T , which below u_{s_0} looks like S_0 ;

(k) $u_{s_k \wedge n} = \text{root } S_k \wedge n$ (note that at this moment we have surely defined $u_{s_{k+1}} \in T$), $S_{k+1} \leq T_k|u_{s_{k+1}}$, and $v_{k+1} \in 2^{<\omega}$ satisfy

$$S_{k+1} \Vdash "[v_0 \wedge \dots \wedge v_k \wedge v_{k+1}] \subseteq \dot{G}_{k+1}, "$$

T_{k+1} is a modification of T_k , which below $u_{s_{k+1}}$ looks like S_{k+1} .

Finally note that the sequence $\langle u_s : s \in \omega^{<\omega} \rangle$ determines a condition $T' \leq T$, and the sequence $\langle v_n : n \in \omega \rangle$ gives a function $c = v_0 \wedge v_1 \wedge v_2 \wedge \dots$ from $2^\omega \cap V$ such that $T' \Vdash "c \in \bigcap_{n \in \omega} G_n."$ $\square$

Remark. Theorem 2.1 shows that, in some sense, $\mathbb{D}_\omega$ is close to the Cohen forcing. Actually, we have proved there that if $r \in \omega^\omega$ is a real that does not belong to any set from $\mathcal{D}_\omega$ coded in V then there is a Cohen real in $V[r]$. Similarly, one can prove that if reals from the ground model V form a set of the first category in an extension, then $\omega^\omega \cap V$ belongs to $\mathcal{D}_\omega$ in this extension. It seems that the forcing $\mathbb{D}_\omega$ is the best one for adding Cohen reals in countable support iterations.

Proposition 2.2. *Suppose $\mathcal{X}$ if finite. Then*

- (a) *If $T \Vdash "\dot{t} \in V," n \in \omega$ then there are $T' \leq_n T$ and $A \in [V]^{\mathcal{X}|^n}$ such that $T' \Vdash "\dot{t} \in A"$;*
- (b) *$\mathbb{D}_\mathcal{X} \Vdash "(\forall A \in \mathbb{L})(\exists B \in \mathbb{L} \cap V)(A \subseteq B),"$ so the forcing with $\mathbb{D}_\mathcal{X}$ does not change the ideal of measure and the ideal of category.*

Proof. (b) Applying (a) and the fusion property of orders $\leq_n$ one can get that $\mathbb{D}_\mathcal{X} \Vdash "(\forall f \in \omega^\omega)(\exists \phi : \omega \rightarrow [\omega]^{<\omega}, \phi \in V)(\forall n \in \omega)(|\phi(n)| = 2^n \ \& \ f(n) \in \phi(n))."$ Hence $\mathbb{D}_\mathcal{X} \Vdash$ "every set of measure zero can be covered by a Borel set of measure zero coded in V "; this implies the analogous property for the ideal $\mathbb{K}$, $\mathbb{D}_\mathcal{X} \Vdash "(\forall A \in \mathbb{K})(\exists B \in \mathbb{K} \cap V)(A \subseteq B)"$ (cf. [F]). $\square$

The above result can be strengthened. Let $k \geq 2$ be a natural number. A tree $T \subseteq \omega^{<\omega}$ is said to be a k -tree if $(\forall s \in T)(|\text{succ}_T(s)| \leq k)$. We say that a notion of forcing $\mathbb{P}$ has the k -localization property if

$$\mathbb{P} \Vdash (\forall f \in \omega^\omega)(\exists T \in V)("T \text{ is a } k\text{-tree on } \omega" \ \& \ f \in [T]).$$

Let us remark that if $\mathbb{P}$ has the k -localization property then, in $V^\mathbb{P}$, reals from $(k+1)^\omega$ can be covered by sets from $\mathcal{D}_{k+1}$ coded in V . Let $\mathbb{S}_k$ be the Silver forcing below k . It contains all partial functions $p: \text{dom } p \rightarrow k$ such that $\text{dom } p \subseteq \omega$ is coinfinite and it is ordered by inclusion.

Theorem 2.3. *Countable support iterations and countable support products of forcings $\mathbb{D}_i$ or $\mathbb{S}_i$ for $i \leq k$ have the k -localization property.*

Proof. We do not find a general proof of this theorem but the proofs of all possible cases are very similar. We present the proof of Theorem 2.3 for countable support product of $\mathbb{D}_k$ because we will use this version.

Recall that $T_1 \leq_n T_2$ if and only if $T_1 \leq T_2$ and $(T_1)^n = (T_2)^n$, where T^n consists of all elements of s of T such that $\text{succ}_T(s) = k$ and there exists exactly $n-1$ initial segments of s with the same property. Note that $|T^n| = k^{n-1}$ for each $n \geq 1$ and $T \in \mathbb{D}_k$. Moreover each function $\sigma \in k^n$ uniquely determines an element $s = s_\sigma \wedge \sigma(n-1)$, where $s_\sigma \in T^n$. We put $(T)_\sigma = T_s$.

Let $\mathbb{P}$ be the countable support product of κ copies of $\mathbb{D}_k$.

For $F \in [\kappa]^{<\omega}$ and $\eta: F \rightarrow \omega$ we define orders $\leq_{F,\eta}$ on $\mathbb{P}$ by $p \leq_{F,\eta} q$ if and only if $p \leq q$ and $(\forall \alpha \in F)(p(\alpha) \leq_{\eta(\alpha)} q(\alpha))$.

Orders $\leq_{F, \eta}$ have the fusion property: Suppose that $p_{n+1} \leq_{F_n, \eta_n} p_n$, $F_n \subseteq F_{n+1}$, $\bigcup_{n \in \omega} F_n = \bigcup_{n \in \omega} \text{supp}(p_n)$ and if $\alpha \in F_n$ then

$$(\forall m \geq n)(\eta_m(\alpha) \leq \eta_{m+1}(\alpha)) \quad \text{and} \quad \lim_{m \rightarrow \infty} \eta_m(\alpha) = \infty.$$

Then there exists $p \in \mathbb{P}$ such that $p \leq_{F_n, \eta_n} p_n$ for every $n \in \omega$.

Let $\sigma \in \prod_{\alpha \in F} k^{\eta(\alpha)}$, $F \in [\kappa]^{<\omega}$, and $\eta: F \rightarrow \omega$. Let $p \in \mathbb{P}_\kappa$. The condition $p^* \sigma$ is defined by $(p^* \sigma)(\alpha) = p(\alpha)$ for $\alpha \notin F$ and if $\alpha \in F$ then $(p^* \sigma)(\alpha) = (p(\alpha))_{\sigma(\alpha)}$. Note that $q \leq_{F, \eta} p$ implies $q^* \sigma \leq p^* \sigma$. Moreover if $p \Vdash \dot{\tau} \in \omega$ then there exists $q \leq_{F, \eta} p$ such that $q^* \sigma$ decides $\dot{\tau}$.

Suppose that $\Vdash \dot{f} \in \omega^\omega$. Let $A \subseteq \omega^{<m}$ be a k -tree. We say that A is $(p, F, \eta, \dot{f})$ -saturated if

- (i) $p \Vdash \dot{f}|m \in A$,
- (ii) $p^* \sigma$ decides $\dot{f}|m$ for every $\sigma \in \prod_{\alpha \in F} k^{\eta(\alpha)}$,
- (iii) if $q \leq_{F, \eta} p$, $m' > m$, and $\sigma_1, \sigma_2 \in \prod_{\alpha \in F} k^{\eta(\alpha)}$ are such that $q^* \sigma_1$ and $q^* \sigma_2$ decide $\dot{f}|m'$ in different ways then they decide $\dot{f}|m$ differently.

Claim. Suppose that we have given $p \in \mathbb{P}$, $F \in [\kappa]^{<\omega}$, $\eta: F \rightarrow \omega$, and a $(p, F, \eta, \dot{f})$ -saturated k -tree $A \subseteq \omega^{\leq m}$. Let $F_1 = F \cup \{\beta\}$, $\eta_1 = \eta \cup \{(\beta, 1)\}$. Then there exists $q \leq_{F_1, \eta_1} p$ and a $(q, F_1, \eta_1, \dot{f})$ -saturated k -tree $A' \subseteq \omega^{\leq M}$ containing A .

Fix an enumeration $\{\sigma_1, \dots, \sigma_z\} = \prod_{\alpha \in F} k^{\eta(\alpha)}$. Let $\sigma_i \wedge j = \sigma_i \cup \{(\beta, \langle j \rangle)\}$. We construct conditions p_i , natural numbers m_i , and $A^i \in [\omega^{<\omega}]^{\leq k}$ for $i \leq z$: $p_1 \leq_{F_1, \eta_1} p$ and $m_1 > m$ are such that $p_1^*(\sigma_1 \wedge j)$ decides $\dot{f}|[m, m_1]$ for each $j < k$ and the set $A^1 = \{s \in \omega^{[m, m_1]} : (\exists j < k)(p_1^*(\sigma_1 \wedge j) \Vdash \dot{f}|[m, m_1] = s)\}$ has the greatest possible cardinality.

$p_{i+1} \leq_{F_1, \eta_1} p_i$ and $m_{i+1} > m_i$ are such that $p_{i+1}^*(\sigma_i \wedge j)$ decides $\dot{f}|[m, m_{i+1}]$ for each $j < k$, $l \leq i+1$ and the set $A^{i+1} = \{s \in \omega^{[m, m_{i+1}]} : (\exists j < k)(p_{i+1}^*(\sigma_{i+1} \wedge j) \Vdash \dot{f}|[m, m_{i+1}] = s)\}$ has the greatest possible cardinality.

Note that if $p_i^*(\sigma_i \wedge j)$, $p_i^*(\sigma_i \wedge j)$ decide $\dot{f}|m$ in the same way for some $i' < i \leq z$ and $j < k$ then they do the same with $\dot{f}|m_1$. Put $M = m_z$, $q = p_z$, and $A' = \{s \in \omega^M : (\exists i \leq z, j < k)(q^*(\sigma_i \wedge j) \Vdash \dot{f}|M = s)\}$. Then A' contains A and it is a k -tree (by the last remark). To show that it is $(q, F_1, \eta_1, \dot{f})$ -saturated we have to verify condition (iii) only. Suppose that $r \leq_{F_1, \eta_1} q$, $M' > M$, and $\sigma, \delta \in \prod_{\alpha \in F} k^{\eta(\alpha)}$, $i, j < k$, are such that $r^*(\sigma \wedge i)$ and $r^*(\delta \wedge j)$ decide $\dot{f}|M'$ in different ways but they decide $\dot{f}|M$ in the same way. We may assume that $r^*(\sigma \wedge j)$ and $r^*(\delta \wedge i)$ decide $\dot{f}|M'$. Then $r^*(\sigma \wedge i)$ and $r^*(\delta \wedge i)$ decide $\dot{f}|M'$ in the same way. Hence $r^*(\delta \wedge i)$ and $r^*(\delta \wedge j)$ decide $\dot{f}|M'$ differently. But it follows from the construction (maximality of A^i) that then they decide $\dot{f}|M$ differently—a contradiction.

Claim. Suppose that we have given $p \in \mathbb{P}$, $F \in [\kappa]^{<\omega}$, $\eta: F \rightarrow \omega$, and a $(p, F, \eta, \dot{f})$ -saturated k -tree $A \subseteq \omega^{\leq m}$. Let $\beta \in F$ and η_1 be a modification of η such that $\eta_1(\beta) = \eta(\beta) + 1$. Then there exist $q \leq_{F, \eta_1} p$ and a $(q, F, \eta_1, \dot{f})$ -saturated k -tree $A' \subseteq \omega^{\leq M}$ containing A .

The proof of this claim can be obtained from the proof of the previous one by redefining $\sigma \wedge j$. Now it should be equal $\sigma|F \setminus \{\beta\} \cup \{(\beta, \sigma(\beta) \wedge j)\}$.

Suppose $p \Vdash \dot{f} \in \omega^\omega$. Apply the claims above to construct $p_n \leq p$, $F_n \in [\kappa]^{<\omega}$, $\eta_n: F_n \rightarrow \omega$ with the fusion property, and increasing k -trees $A_n \subseteq \omega^{\leq m_n}$ such that $m_n < m_{n+1}$ and $p_n \Vdash \dot{f} \restriction m_n \in A_n$. Let $A = \bigcup_{n \in \omega} A_n$ and let q be the fusion of $\langle p_n : n \in \omega \rangle$. Then A is a k -tree and $q \Vdash \dot{f} \in [A]$. $\square$

Since obviously forcings $\mathbb{D}_{k+1}$ and $\mathbb{S}_{k+1}$ do not have the k -localization property forcings with this property, add neither $\mathbb{D}_{k+1}$ nor $\mathbb{S}_{k+1}$ -generic reals. Thus we get

Corollary 2.4. *No countable support iteration of forcings $\mathbb{S}_2$ adds an $\mathbb{S}_4$ -generic real.*

3. CARDINAL COEFFICIENTS

For an ideal $\mathbb{I}$ of subsets of a space X the following cardinal invariants of $\mathbb{I}$, we introduce

$$\begin{aligned} \text{add}(\mathbb{I}) &= \min\{|\mathcal{H}| : \mathcal{H} \subseteq \mathbb{I} \text{ \& } \bigcup \mathcal{H} \notin \mathbb{I}\}, \\ \text{cov}(\mathbb{I}) &= \min\{|\mathcal{H}| : \mathcal{H} \subseteq \mathbb{I} \text{ \& } \bigcup \mathcal{H} = X\}, \\ \text{non}(\mathbb{I}) &= \min\{|\mathcal{H}| : \mathcal{H} \subseteq X \text{ \& } \bigcap \mathcal{H} \notin \mathbb{I}\}, \\ \text{cof}(\mathbb{I}) &= \min\{|\mathcal{H}| : \mathcal{H} \subseteq \mathbb{I} \text{ \& } (\forall A \in \mathbb{I})(\exists B \in \mathcal{H})(A \subseteq B)\}. \end{aligned}$$

The cardinal coefficients of ideals $\mathbb{L}(2^\omega)$ and $\mathbb{K}(2^\omega)$, together with the unbounded number $\mathfrak{b}$ and the dominating number $\mathfrak{d}$, are collected in the so-called Cichoń Diagram (cf. [F, V]).

Theorem 3.1. (a) $\text{cov}(\mathcal{D}_{\mathcal{X}}) \geq \text{cov}(\mathcal{D}_{\mathcal{Y}})$ and $\text{non}(\mathcal{D}_{\mathcal{X}}) \leq \text{non}(\mathcal{D}_{\mathcal{Y}})$, provided $\mathcal{X} \subseteq \mathcal{Y}$.

(b) If $\mathcal{X}$ is finite then $\text{cov}(\mathcal{D}_{\mathcal{X}}) \geq \max\{\text{cov}(\mathbb{K}), \text{cov}(\mathbb{L})\}$ and $\text{non}(\mathcal{D}_{\mathcal{X}}) \leq \min\{\text{non}(\mathbb{K}), \text{non}(\mathbb{L})\}$.

(c) $\text{cov}(\mathcal{D}_\omega) = \text{cov}(\mathbb{K})$ and $\text{non}(\mathcal{D}_\omega) = \text{non}(\mathbb{K})$.

Proof. (a) Let $\phi: \mathcal{Y}^\omega \rightarrow \mathcal{X}^\omega$ be a function given by Proposition 1.3(a). Then $\phi[A] \notin \mathcal{D}_{\mathcal{X}}$ provided $A \notin \mathcal{D}_{\mathcal{Y}}$. If $A_\alpha \in \mathcal{D}_{\mathcal{Y}} (\alpha < \kappa)$, $\bigcup_{\alpha < \kappa} A_\alpha = \mathcal{Y}^\omega$ then $\phi^{-1}[A_\alpha] \in \mathcal{D}_{\mathcal{X}}$ and $\bigcup_{\alpha < \kappa} \phi^{-1}[A_\alpha] = \mathcal{Y}^\omega$.

(b) follows from Theorem 1.1(b).

(c) was proved in [R1], where Bartoszyński's characterization of $\text{non}(\mathbb{K})$ and $\text{cov}(\mathbb{K})$ was applied. Note, however, that we can apply Proposition 1.3(b) to prove, similarly as in (a), that $\text{non}(\mathcal{D}_\omega) \geq \text{non}(\mathbb{K})$ and $\text{cov}(\mathcal{D}_\omega) \leq \text{cov}(\mathbb{K})$. Converse inequalities follows from Theorem 1.1(b). $\square$

Theorem 3.2. *There exists a family $F \subseteq \mathcal{D}_{\mathcal{X}}$ of the cardinality $\mathfrak{c}$ such that for every uncountable $G \subseteq F$, $\bigcup G \notin \mathcal{D}_{\mathcal{X}}$.*

Proof. Let $\{c_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathcal{X}^\omega$ be such a family that $\alpha < \beta < \mathfrak{c}$ implies $(\forall n)(\exists m > n)(c_\alpha(m) \neq c_\beta(m))$. For $\alpha < \mathfrak{c}$ put $A_\alpha = \{c \in \mathcal{X}^\omega : (\forall n)(c(n) \neq c_\alpha(n))\}$. Clearly $A_\alpha \in \mathcal{D}_{\mathcal{X}}$. Suppose that $R \subseteq \mathfrak{c}$ is uncountable. We want to show that $\bigcup_{\alpha \in R} A_\alpha \notin \mathcal{D}_{\mathcal{X}}$. Suppose not and let $f: \mathcal{X}^{<\omega} \rightarrow \mathcal{X}$ witness that $\bigcup_{\alpha \in R} A_\alpha \in \mathcal{D}_{\mathcal{X}}$. For $\alpha < \mathfrak{c}$ and $s \in \mathcal{X}^{<\omega}$, let $B_\alpha(s) = \{t \in \mathcal{X}^{<\omega} : s \subseteq t \text{ \& } (\forall n \in \text{dom } t \setminus \text{dom } s)(t(n) \neq c_\alpha(n))\}$.

Claim. $(\exists \alpha \in R)(\forall s \in \mathcal{X}^{<\omega})(\exists t \in B_\alpha(s))(f(t) \neq c_\alpha(lh(t)))$. If not, we would find distinct $\alpha, \beta \in R$ and $s \in \mathcal{X}^{<\omega}$ such that $(\forall t \in B_\alpha(s))(f(t) = c_\alpha(lh(t)))$ and $(\forall t \in B_\beta(s))(f(t) = c_\beta(lh(t)))$. Let $n \geq lh(s)$ be the first such that

$c_\alpha(n) \neq c_\beta(n)$. Choose $t \in \mathcal{X}^n$ such that $s \subseteq t$ and $(\forall k \in n \setminus \text{dom } s)(t(k) \neq c_\alpha(k) = c_\beta(k))$. Then $t \in B_\alpha(s) \cap B_\beta(s)$ and hence $c_\alpha(n) = f(t) = c_\beta(n)$ —a contradiction.

Thus let $a \in R$ be given by the claim. Construct inductively a sequence $\langle t_n : n \in \omega \rangle \subseteq \mathcal{X}^{<\omega}$ such that $t_0 \in B_\alpha(\emptyset)$ and $f(t_0) \neq c_\alpha(lh(t_0))$, $t_{n+1} \in B_\alpha(t_n \frown f(t_n))$, and $f(t_{n+1}) \neq c_\alpha(lh(t_{n+1}))$. Put $d = \bigcup_{n \in \omega} t_n$. Then evidently $d \in A_\alpha \setminus \{c \in \mathcal{X}^\omega : (\exists n \in \omega)(\forall m > n)(c(m) \neq f(c \restriction m))\}$, and hence f cannot witness that $\bigcup_{\alpha \in R} A_\alpha \in \mathcal{D}_\mathcal{X}$. $\square$

Corollary 3.3. $\text{add}(\mathcal{D}_\mathcal{X}) = \omega_1$ and $\text{cof}(\mathcal{D}_\mathcal{X}) = 2^\omega$.

Putting together Corollary 3.3 and Theorems 3.1(c) we get the full description of the cardinal coefficients of $\mathcal{D}_\omega$: $\text{add}(\mathcal{D}_\omega) = \omega_1$, $\text{non}(\mathcal{D}_\omega) = \text{non}(\mathbb{K})$, $\text{cov}(\mathcal{D}_\omega) = \text{cov}(\mathbb{K})$, and $\text{cof}(\mathcal{D}_\omega) = \mathfrak{c}$. Below we give some consistency results concerning the coefficients non and cov of $\mathcal{D}_\mathcal{X}$ for finite $\mathcal{X}$.

Theorem 3.4. (a) $\text{CON}(\text{ZFC} + (\forall k \geq 2)(\text{cov}(\mathcal{D}_k) = \omega_2 = \mathfrak{c}) + \text{cof}(\mathbb{L}) = \omega_1)$.

(b) Suppose that $k \geq 2$. Then $\text{CON}(\text{ZFC} + \text{cov}(\mathcal{D}_{k+1}) = \text{cof}(\mathbb{L}) = \omega_1 + \text{cov}(\mathcal{D}_k) = \omega_2 = \mathfrak{c})$.

Proof. (a) Suppose that $V \models \text{CH}$. Let $\mathbb{P} = \prod_{\alpha < \kappa} \mathbb{P}_\alpha$ be the countable support product of forcings $\mathbb{D}_k$ ($k \geq 2$) such that $|\{\alpha < \kappa : \mathbb{P}_\alpha = \mathbb{D}_k\}| = \omega_2$ for each $k \geq 2$. Standard arguments (see [B]) show that $\mathbb{P}$ preserves cardinal numbers and $\mathbb{P} \Vdash \mathfrak{c} = \omega_2$. Moreover, if $F \in [\omega_2]^{<\omega}$ is such that for $\alpha \in F$, $\mathbb{P}_\alpha = \mathbb{D}_m$ for some $m \leq n$ and $p \Vdash \mathfrak{c} \in \omega$ then there exist $q \leq_F, n p$ and $A \subseteq \omega$, $|A| \leq n^{n^n}$ such that $q \Vdash \mathfrak{c} \in A$. Hence by fusion arguments one can prove that $\mathbb{P} \Vdash (\forall f \in \omega^\omega)(\exists \phi : \omega \rightarrow [\omega]^{<\omega}, \phi \in V)(\forall n \in \omega)(f(n) \in \phi(n) \& |\phi(n)| \leq n^{n^n})$ and consequently $\mathbb{P} \Vdash (\forall A \in \mathbb{L})(\exists B \in \mathbb{L} \cap V)(A \subseteq B)$. Hence $\mathbb{P} \Vdash \text{cof}(\mathbb{L}) = \omega_1$ (which in particular, means that $\Vdash_{\omega_2} \text{non}(\mathcal{D}_\mathcal{X}) = \omega_1$ for each finite $\mathcal{X}$). Let $k \geq 2$.

Claim. $\mathbb{P} \Vdash \text{cov}(\mathcal{D}_k) = \omega_2$.

First note that if $f : k^{<\omega} \rightarrow k$ and c is a $(\mathbb{D}_k)^V$ -generic real over $V[f]$ then infinitely often $f(c \restriction m) = c(m)$. Let $\langle c_\alpha : \alpha < \kappa \rangle$ be a $\mathbb{P}$ -generic sequence over V . Suppose now that, in $V[\langle c_\alpha : \alpha < \kappa \rangle]$, for each $\alpha < \omega_1$ we have given a function $f_\alpha : k^{<\omega} \rightarrow k$. Let $\dot{f}_\alpha$ be a $\mathbb{P}$ -name for f_α . Each $\dot{f}_\alpha$ is a $\prod_{\beta < \beta(\alpha)} \mathbb{P}_\beta$ -name for some $\beta(\alpha) < \omega_2$. Choose $\beta < \omega_2$ such that $\mathbb{P}_\beta = \mathbb{D}_k$ and $\beta(\alpha) < \beta$ for each $\alpha < \omega_1$. Thus $\langle \dot{f}_\alpha : \alpha < \omega_1 \rangle$ is a sequence $\prod_{\alpha < \beta} \mathbb{P}_\alpha$ -names. Since c_β is $\mathbb{D}_k$ -generic over $V[\langle c_\alpha : \alpha < \beta \rangle]$ we get that, in $V[\langle c_\alpha : \alpha < \kappa \rangle]$, for each $\alpha < \omega_1$ infinitely often $f_\alpha(c_\beta \restriction m) = c_\beta(m)$. Consequently, sets from $\mathcal{D}_k$ determined in $V[\langle c_\alpha : \alpha < \kappa \rangle]$ by functions f_α do not cover k^ω .

(b) Fix $k \geq 2$. Suppose $V \models \text{CH}$. Let $\mathbb{P}$ be a countable support product of ω_2 copies of forcings $\mathbb{D}_k$. Then $\mathbb{P}$ preserves cardinal numbers and $\mathbb{P} \Vdash \text{cof}(\mathbb{L}) = \omega_1$ & $\mathfrak{c} = \omega_2$. By Theorem 2.3, $\mathbb{P}$ has the k -localization property. Thus, in $V^\mathbb{P}$, $(k+1)^\omega$ can be covered by sets from $\mathcal{D}_{k+1}$ coded in V . Consequently $\mathbb{P} \Vdash \text{cov}(\mathcal{D}_{k+1}) = \omega_1$. As in (a) one can prove that $\mathbb{P} \Vdash \text{cov}(\mathcal{D}_k) = \omega_2$. $\square$

Theorem 3.5. $\text{CON}(\text{ZFC} + (\forall k \geq 3)(\text{non}(\mathcal{D}_k) = \text{cov}(\mathbb{K}) = \omega_2))$.

Proof. Let $\mathbb{P}$ consist of all pairs (T, n) such that T is a 2-tree on ω , $||T|| < \omega$, and $|\text{succ}_T(s)| = 1$ for each $s \in T$ of length $\geq n$. We order $\mathbb{P}$ by $(T, n) \leq (S, m)$ if and only if $S \subseteq T$, $n \geq m$, and $T \restriction m = S \restriction m$.

Claim. $\mathbb{P}$ has the ccc property.

Suppose that $(T_\alpha, n_\alpha) \in \mathbb{P}$ for $\alpha < \omega_1$. We may assume that $n_\alpha = n$ and $T_\alpha|n = S$ for every $\alpha < \omega_1$. Let $\alpha < \beta < \omega_1$. Put $T = T_\alpha \cup T_\beta$. Then $T|n = S$ and $s \in T$, $lh(s) \geq n$ imply that s has exactly one successor in each T_α and T_β . Hence T is a 2-tree, $||T|| < \omega$. Choose $m > n$ such that for each $s \in T$, $lh(s) \geq m$ there is one successor of s in T only. Then $(T, m) \in \mathbb{P}$ and it lies below (T_α, n) and (T_β, n) .

Claim. $\mathbb{P} \Vdash "\omega^\omega \cap V$ can be covered by countably many 2-trees on ω " and consequently $\mathbb{P} \Vdash "(\forall k \geq 3)(k^\omega \cap V \in \mathcal{D}_k)." "$

Let $G \subseteq \mathbb{P}$ be a generic set over V , $T_G = \bigcup \{T : (\exists n)((T, n) \in G)\}$. Clearly T_G is a 2-tree on ω . For $s \in \omega^n$ and $t \in T_G \cap \omega^n$, put $T_{s,t} = \{r \in \omega^{<\omega} : r \subseteq s \text{ or } (\exists t' \in \omega^{<\omega})(r = s \wedge t' \text{ \& } t \wedge t' \in T)\}$. Obviously $T_{s,t}$ are 2-trees. Easy density arguments show that, in $V[G]$, $\omega^\omega \cap V \subseteq \bigcup \{[T_{s,t}] : n \in \omega, s \in \omega^n, t \in T_G \cap \omega^n\}$.

Suppose that $V \models \text{CH}$. Let $\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \omega_2 \rangle$ be a finite support iteration of forcings $\mathbb{P}$. Then $\mathbb{P}_{\omega_2}$ has the ccc property and $\Vdash_{\omega_2}$ "for each $\alpha < \omega_2$, $\omega^\omega \cap V^{\mathbb{P}_\alpha}$ can be covered by countably many 2-trees on ω ." Since, in $V^{\mathbb{P}_{\omega_2}}$, each subset of ω^ω of the cardinality ω_1 is included in $\omega^\omega \cap V^{\mathbb{P}_\alpha}$ for some $\alpha < \omega_2$ we get $\Vdash_{\omega_2} "(\forall k \geq 3)(\text{non}(\mathcal{D}_k) = \omega_2)." "$ Because of Cohen reals added at limit stages we have $\Vdash_{\omega_2} "\text{cov}(\mathbb{K}) = \omega_2"$ (hence $\Vdash_{\omega_2} "(\forall k \geq 2)(\text{cov}(\mathcal{D}_k) = \omega_2)"$). $\square$

Note that in models presented in Theorems 3.4 and 3.5 we have $\text{cov}(\mathcal{D}_k) \geq \text{non}(\mathcal{D}_k)$ for each $k \geq 3$. Thus the following question seems to be natural.

Problem 3.6. Suppose that $\mathcal{X}$ is finite (but $|\mathcal{X}| \geq 3$). Can $\text{non}(\mathcal{D}_{\mathcal{X}})$ be greater than $\text{cov}(\mathcal{D}_{\mathcal{X}})$?

4. COMPACT SETS FROM THE IDEAL

Let $\mathcal{K}(\mathcal{X}^\omega)$ denote the space of all compact subsets of $\mathcal{X}^\omega$, equipped with the Vietoris topology generated by the subbase consisting of the sets $U(G) = \{F \in \mathcal{K}(\mathcal{X}^\omega) : F \subseteq G\}$, $V(G) = \{F \in \mathcal{K}(\mathcal{X}^\omega) : F \cap G \neq \emptyset\}$ where G denotes an open subset of $\mathcal{X}^\omega$. It is well known that $\mathcal{K}(\mathcal{X}^\omega)$ is a perfect Polish space that is compact if $\mathcal{X}$ is finite (cf. [K]).

Some σ -ideals have recently been studied from the point of view of the structure of their traces on compact sets. Kechris, Louveau, and Woodin showed that the trace of each σ -ideal on compact sets is either very simple (Π_2^0) or at least Π_1^1 (cf. [KLW]). Our next result shows that for the ideal $\mathcal{D}_{\mathcal{X}}$ the second possibility takes place.

Theorem 4.1. Let $\mathcal{X}$ be finite. Then $\mathcal{D}_{\mathcal{X}} \cap \mathcal{K}(\mathcal{X}^\omega) \in \Pi_1^1 \setminus \Sigma_1^1$ and it is of the first category in $\mathcal{K}(\mathcal{X}^\omega)$.

Proof. Since $\mathcal{X}^{<\omega}$ is countable the space $\mathcal{P}$ of all subsets of $\mathcal{X}^{<\omega}$ with the obvious product topology is homeomorphic with 2^ω . Trees form the closed subset of this space. For $A \in \mathcal{K}(\mathcal{X}^\omega)$ we have

$$A \notin \mathcal{D}_{\mathcal{X}} \equiv (\exists T \in \mathcal{P})(\text{"}T \text{ is a tree" \& "}T \text{ satisfies (**)" \& } [T] \subseteq A).$$

The formula " T is a tree and T satisfies (**)" presents a Π_2^0 subset of $\mathcal{P}$. Moreover, $[T] \subseteq A \equiv (\forall x \in \mathcal{X}^\omega)(x \in A \vee (\exists n \in \omega)(x|n \notin T))$. The formula

$x \in A \vee (\exists n \in \omega)(x|n \notin T)$ describes a Π_2^0 subset of $\mathcal{P} \times \mathcal{K}(\mathcal{X}^\omega) \times \mathcal{X}^\omega$. Since $\mathcal{X}^\omega$ is compact, the $\forall \mathcal{X}^\omega$ -projection of a Π_2^0 set is of the class Π_2^0 . Hence the complement of $\mathcal{D}_{\mathcal{X}} \cap \mathcal{K}(\mathcal{X}^\omega)$ is analytic in $\mathcal{K}(\mathcal{X}^\omega)$. It follows from Theorem 6.4 of [R2] that $\mathcal{D}_{\mathcal{X}} \cap \mathcal{K}(\mathcal{X}^\omega)$ is of the first category. Since $\mathcal{D}_{\mathcal{X}} \cap \mathcal{K}(\mathcal{X}^\omega)$ is dense (it contains all finite sets) and it is of the first category, it cannot be Π_2^0 . Hence $\mathcal{D}_{\mathcal{X}} \cap \mathcal{K}(\mathcal{X}^\omega) \notin \Sigma_1^1$ (cf. [KLW]). $\square$

For $A \subseteq \mathcal{X}^\omega \times \mathcal{X}^\omega$ and $d \in \mathcal{X}^\omega$, let $A_d = \{c \in \mathcal{X}^\omega : (d, c) \in A\}$. Since every set from $\text{BOREL}(\mathcal{X}^\omega) \setminus \mathcal{D}_{\mathcal{X}}$ contains a compact set that does not belong to $\mathcal{D}_{\mathcal{X}}$, Theorem 4.1 can be applied to establish the complexity of the class of sets of the form $\{c \in \mathcal{X}^\omega : A_c \notin \mathcal{D}_{\mathcal{X}}\}$. Let $\Phi: \mathcal{P}(\mathcal{X}^\omega \times \mathcal{X}^\omega) \rightarrow \mathcal{P}(\mathcal{X}^\omega)$ be given by $\Phi(A) = \{c \in \mathcal{X}^\omega : A_c \notin \mathcal{D}_{\mathcal{X}}\}$. Then easy calculations and an application of Theorem 4.1 show

Corollary 4.2. (a) $\Phi[\Sigma_1^0(\mathcal{X}^\omega \times \mathcal{X}^\omega)] = \Sigma_1^0(\mathcal{X}^\omega)$.

(b) $\Phi[\Pi_1^0(\mathcal{X}^\omega \times \mathcal{X}^\omega)] = \Phi[\Sigma_3^0(\mathcal{X}^\omega \times \mathcal{X}^\omega)] = \Sigma_1^1(\mathcal{X}^\omega)$.

(c) $\Phi[\text{BOREL}(\mathcal{X}^\omega \times \mathcal{X}^\omega)] \subseteq \Sigma_2^1(\mathcal{X}^\omega)$.

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