FINITE SUBSTRUCTURE LATTICES OF MODELS OF PEANO ARITHMETIC

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ABSTRACT. Some new finite lattices (for example, M_4 , M_7 , and the hexagon lattice) are shown to be isomorphic to the lattice of elementary substructures of a model of Peano Arithmetic.

The set of elementary substructures of a model $\mathcal{N}$ of Peano Arithmetic forms a lattice $\text{Lt}(\mathcal{N})$, the substructure lattice of $\mathcal{N}$. It is unknown whether there are finite lattices that are not isomorphic to any substructure lattice. Indeed, it was conjectured in [8] that for any finite lattice L there is some $\mathcal{N} \models PA$ for which $L \cong \text{Lt}(\mathcal{N})$. It was proved in [8] that if $2 \leq n < \omega$, then there is $\mathcal{N}$ such that $\text{Lt}(\mathcal{N}) \cong \Pi(n)$. (The partition lattices $\Pi(n)$ are defined below.) For $3 \leq n < \omega$, the lattice M_n is the unique lattice that has $n + 2$ elements and n atoms. Thus, the lattice M_3 , which is isomorphic to $\Pi(3)$, is isomorphic to a substructure lattice, although it is known from [2] or [4] that if $\mathcal{N}$ is a model of True Arithmetic and $3 \leq n < \omega$, then $\text{Lt}(\mathcal{N}) \not\cong M_n$. Other examples of finite lattices that are not substructure lattices of models of True Arithmetic are given in [8]. The lattices M_n for $4 \leq n < \omega$ and the hexagon lattice were identified in [8] as specific finite lattices that were not known to be isomorphic to substructure lattices. It is a consequence of the theorem in this note that M_n , whenever $n = p^a + 1$ for some prime p , the hexagon lattice, and the lattice M_7 are isomorphic to substructure lattices. It is still unknown whether or not M_{11} is isomorphic to a substructure lattice.

All previously known positive results about finite substructure lattices are contained in [8]. Prior to [8], Paris [4] proved that all finite distributive lattices occur as substructure lattices. (Also see [7].) Wilkie [9] showed that the pentagon lattice is a substructure lattice, and Paris [5] proved that M_3 is a sublattice of an infinite substructure lattice.

More generally, intermediate structure lattices will be considered. For $\mathcal{M} \prec \mathcal{N} \models PA$, let $\text{Lt}(\mathcal{N}/\mathcal{M}) = \{\mathcal{M}_0 \in \text{Lt}(\mathcal{N}) : \mathcal{M} \preceq \mathcal{M}_0 \preceq \mathcal{N}\}$, regarded as a sublattice of $\text{Lt}(\mathcal{N})$. Notice that if $\mathcal{M}$ is a minimal model of PA and $\mathcal{M} \prec \mathcal{N}$, then $\text{Lt}(\mathcal{N}) = \text{Lt}(\mathcal{N}/\mathcal{M})$.

Given a set A , let $\Pi(A)$ be the set of all partitions of A . If $a, b \in A$ and $\pi \in \Pi(A)$, then we write $a \approx b \pmod{\pi}$ if $\{a, b\} \subseteq C$ for some $C \in \pi$.

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There are two extreme partitions $\mathbf{0}_A$ and $\mathbf{1}_A$ in $\Pi(A)$, where

$$\begin{aligned} a \approx b \pmod{\mathbf{1}_A} & \text{ iff } a = b, \\ a \approx b \pmod{\mathbf{0}_A} & \text{ for every } a, b \in A. \end{aligned}$$

We partially order $\Pi(A)$ so that if $\pi_1, \pi_2 \in \Pi$ then $\pi_1 \leq \pi_2$ iff π_2 is a refinement of π_1 . In particular, $\mathbf{0}_A \leq \pi \leq \mathbf{1}_A$ for each $\pi \in \Pi(A)$. With this partial order, $\Pi(A)$ becomes a complete lattice (which is the dual of the lattice usually considered). Thus, if $\pi_1, \pi_2 \in \Pi(A)$ then $\pi_1 \vee \pi_2 = \{C_0 : C_0 = C_1 \cap C_2 \neq \emptyset \text{ for some } C_1 \in \pi_1 \text{ and } C_2 \in \pi_2\}$.

Let $(A, \mathcal{F})$ be an algebra (in the sense of universal algebra); that is, each $f \in \mathcal{F}$ is a function $f: A^k \rightarrow A$ for some $k < \omega$. We let $\Pi(A, \mathcal{F})$ be the sublattice of $\Pi(A)$ consisting of all partitions of A that correspond to congruences of $(A, \mathcal{F})$. Thus, $\pi \in \Pi(A, \mathcal{F})$ iff whenever $f \in \mathcal{F}$ is k -ary and $a_0, a_1, \dots, a_{k-1}, b_0, b_1, \dots, b_{k-1} \in A$ are such that $a_i \approx b_i \pmod{\pi}$ for each $i < k$, then

$$f(a_0, a_1, \dots, a_{k-1}) \approx f(b_0, b_1, \dots, b_{k-1}) \pmod{\pi}.$$

The theorem can now be stated.

Theorem. *Let $\mathcal{M} \models PA$ be countable and nonstandard, and let $(A, \mathcal{F})$ be a finite algebra. Then $\mathcal{M}$ has a cofinal elementary extension $\mathcal{N}$ such that $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \Pi(A, \mathcal{F})$.*

We need some definitions taken from [8].

Let $L = (L, \wedge, \vee)$ be a finite lattice. We denote the minimum and maximum elements of L by 0 and 1 . A *representation* α of L in $\Pi(A)$ is an injection $\alpha: L \rightarrow \Pi(A)$ such that $\alpha(0) = \mathbf{0}_A$, $\alpha(1) = \mathbf{1}_A$ and $\alpha(x \vee y) = \alpha(x) \vee \alpha(y)$ for each $x, y \in L$. Notice that we do not require that a representation satisfy the identity $\alpha(x \wedge y) = \alpha(x) \wedge \alpha(y)$.

Let $\alpha: L \rightarrow \Pi(A)$ be a representation. Then α is *finite* if A is finite, and α is *nontrivial* if $|\alpha(x)| \neq 2$ whenever $x \in L$. The representation $\alpha: L \rightarrow \Pi(A)$ is a *congruence* representation if there is an algebra $(A, \mathcal{F})$ such that $\alpha: L \rightarrow \Pi(A, \mathcal{F})$ is a lattice isomorphism. For $n \geq 1$ let $\alpha^n: L \rightarrow \Pi(A^n)$ be the representation such that for each $c, d \in A^n$ and $x \in L$,

$$c \approx d \pmod{\alpha^n(x)} \Leftrightarrow \text{for every } i < n, c_i \approx d_i \pmod{\alpha(x)}.$$

If we let $\mathcal{F}_1$ be the set of unary polynomial functions of $(A, \mathcal{F})$, then $\Pi(A, \mathcal{F}_1) = \Pi(A, \mathcal{F})$. Therefore, when considering some $\Pi(A, \mathcal{F})$, we can assume that $\mathcal{F}$ consists only of unary functions.

It is known [3] that every finite lattice has a congruence representation. However, it is an open question of some apparent difficulty whether every finite lattice has a finite congruence representation. The lattice M_{11} is a specific lattice that has no known finite congruence representation. We present some examples of lattices that have finite congruence representations.

Let L be a finite distributive lattice. Let $A = \{x \in L : x = 1 \text{ or } x \text{ is a meet-irreducible}\}$, considered as a subposet of L . Let $\mathcal{F}$ be the set of all functions $f: A \rightarrow A$ such that $f(a) \geq a$ for each $a \in A$. Then L and $\Pi(A, \mathcal{F})$ are isomorphic lattices. In fact, let $\alpha: L \rightarrow \Pi(A)$ be such that for each $x \in L$, $\alpha(x)$ is the partition whose only equivalence class that (possibly) is not a singleton is $\{a \in A : a \geq x\}$. Then $\alpha: L \rightarrow \Pi(A, \mathcal{F})$ is a lattice isomorphism.

Let F be a finite field of order q , and let $n = q + 1$. Let $A = F \times F$, and let $f \in \mathcal{F}$ iff there are $a, b, c, d \in F$ such that $f(x, y) = (ax + b, cy + d)$. Then $\Pi(A, \mathcal{F}) \cong M_n$.

An observation due to Feit [1] implies that M_7 has a finite congruence representation.

Finally, let $A = \{0, 1, 2, 3, 4, 5, 6\}$. Then there is $\mathcal{F}$ such that $\Pi(A, \mathcal{F})$ is isomorphic to the hexagon lattice. Specifically, $\Pi(A, \mathcal{F}) = \{\mathbf{0}_A, \mathbf{1}_A, \{\{0\}, \{1, 2\}, \{3, 4\}, \{5, 6\}\}, \{\{0, 1, 2\}, \{3, 4\}, \{5, 6\}\}, \{\{0\}, \{1, 3\}, \{4, 6\}, \{2, 5\}\}, \{\{0, 4, 6\}, \{1, 3\}, \{2, 5\}\}\}$.

We now proceed with the proof of the theorem.

Lemma 1. Suppose L is a finite lattice, $\alpha: L \rightarrow \Pi(A)$ is a representation, and $2 \leq n < \omega$. Then

- (1) α^n is a nontrivial representation of L ;
- (2) if α is a finite representation then so is α^n ;
- (3) if α is a congruence representation then so is α^n .

Proof. (1) Clearly, if $x \in L$ then $|\alpha^n(x)| = |\alpha(x)|^n \neq 2$.

(2) If A is finite then so is A^n .

(3) Let $(A, \mathcal{F})$ be a unary algebra such that $\alpha: L \rightarrow \Pi(A, \mathcal{F})$ is a lattice isomorphism. Without loss of generality assume that $\mathcal{F}$ has the identity function and all constant functions as members. Let $\mathcal{G}_0$ be the set of all functions $g: A^n \rightarrow A^n$ for which there are $f_0, f_1, \dots, f_{n-1} \in \mathcal{F}$ such that $g(\langle a_0, a_1, \dots, a_{n-1} \rangle) = \langle f_0(a_0), f_1(a_1), \dots, f_{n-1}(a_{n-1}) \rangle$. We introduce the following notation to be used for the rest of this proof: for any x we denote by $\bar{x}$ the n -tuple $\langle x, x, \dots, x \rangle$. Now let $\mathcal{G}_1$ be the set of all functions $g: A^n \rightarrow A^n$ where for some $i < n$, $g(\langle a_0, a_1, \dots, a_{n-1} \rangle) = \bar{a}_i$. Let $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1$. We claim that $\alpha^n: L \rightarrow \Pi(A^n, \mathcal{G})$ is a lattice isomorphism.

First we show that $\alpha^n(x) \in \Pi(A^n, \mathcal{G})$ for $x \in L$. Suppose $g \in \mathcal{G}_0$, so that $g(\langle a_0, a_1, \dots, a_{n-1} \rangle) = \langle f_0(a_0), f_1(a_1), \dots, f_{n-1}(a_{n-1}) \rangle$. Then

$$\begin{aligned} c \approx d \pmod{\alpha^n(x)} &\Rightarrow \text{for all } i < n, c_i \approx d_i \pmod{\alpha(x)} \\ &\Rightarrow \text{for all } i < n, f_i(c_i) \approx f_i(d_i) \pmod{\alpha(x)} \\ &\Rightarrow \langle f_0(c_0), \dots, f_{n-1}(c_{n-1}) \rangle \\ &\quad \approx \langle f_0(d_0), \dots, f_{n-1}(d_{n-1}) \rangle \pmod{\alpha^n(x)} \\ &\Rightarrow g(c) \approx g(d) \pmod{\alpha^n(x)}. \end{aligned}$$

Next suppose $g \in \mathcal{G}_1$, so that $g(\langle a_0, a_1, \dots, a_{n-1} \rangle) = \bar{a}_i$. Then

$$\begin{aligned} c \approx d \pmod{\alpha^n(x)} &\Rightarrow c_i \approx d_i \pmod{\alpha(x)} \\ &\Rightarrow \bar{c}_i \approx \bar{d}_i \pmod{\alpha^n(x)} \\ &\Rightarrow g(c) \approx g(d) \pmod{\alpha^n(x)}. \end{aligned}$$

Therefore, for all $g \in \mathcal{G}$, if $c \approx d \pmod{\alpha^n(x)}$ then $g(c) \approx g(d) \pmod{\alpha^n(x)}$. Hence, $\alpha^n(x) \in \Pi(A^n, \mathcal{G})$.

Secondly, we show that if $\pi \in \Pi(A^n, \mathcal{G})$ then $\pi = \alpha^n(x)$ for some $x \in L$. Let $\pi \in \Pi(A^n, \mathcal{G})$, and let $\rho \in \Pi(A)$ be such that

$$a \approx b \pmod{\rho} \Leftrightarrow \bar{a} \approx \bar{b} \pmod{\pi}.$$

To show that $\rho \in \Pi(A, \mathcal{F})$, consider arbitrary $f \in \mathcal{F}$. Then

$$\begin{aligned} a \approx b \pmod{\rho} &\Rightarrow \bar{a} \approx \bar{b} \pmod{\pi} \\ &\Rightarrow \overline{f(a)} \approx \overline{f(b)} \pmod{\pi} \\ &\Rightarrow f(a) \approx f(b) \pmod{\rho}. \end{aligned}$$

Thus, $\rho \in \Pi(A, \mathcal{F})$ so that $\rho = \alpha(x)$ for some $x \in L$.

We claim that $\alpha^n(x) = \pi$. To see that $\alpha^n(x) \leq \pi$, notice that

$$\begin{aligned} c \approx d \pmod{\pi} &\Rightarrow \text{for each } i < n, \bar{c}_i \approx \bar{d}_i \pmod{\pi} \\ &\Rightarrow \text{for each } i < n, c_i \approx d_i \pmod{\alpha(x)} \\ &\Rightarrow c \approx d \pmod{\alpha^n(x)}. \end{aligned}$$

To show that $\pi \leq \alpha^n(x)$, consider $c, d \in A^n$ for which $c \approx d \pmod{\alpha^n(x)}$. Let $g_0, g_1, \dots, g_{n-1} \in \mathcal{G}_0$ be such that for $i < n$, $g_i(\langle a_0, a_1, \dots, a_{n-1} \rangle) = \langle b_0, b_1, \dots, b_{n-1} \rangle$, where

$$b_j = \begin{cases} d_j & \text{if } j < i, \\ a_j & \text{if } j = i, \\ c_j & \text{if } j > i \end{cases}$$

for all $\langle a_0, a_1, \dots, a_{n-1} \rangle \in A^n$. Then $c = g_0(\bar{c}_0)$, $g_i(\bar{c}_i) \approx g_i(\bar{d}_i) \pmod{\pi}$, $g_i(\bar{d}_i) = g_{i+1}(\bar{c}_{i+1})$, and $g_{n-1}(\bar{d}_{n-1}) = d$. Therefore, $c \approx d \pmod{\pi}$. $\square$

Definition. Let $\alpha: L \rightarrow \Pi(A)$ and $\beta: L \rightarrow \Pi(B)$ be representations of L . Then α *arrows* β (in symbols: $\alpha \rightarrow \beta$) if whenever $\pi \in \Pi(A)$ there is an injection $\theta: B \rightarrow A$ such that

(1) whenever $x \in L$ and $a, b \in B$, then

$$a \approx b \pmod{\beta(x)} \Leftrightarrow \theta(a) \approx \theta(b) \pmod{\alpha(x)}$$

and

(2) there is $x \in L$ such that whenever $a, b \in B$, then

$$a \approx b \pmod{\beta(x)} \Leftrightarrow \theta(a) \approx \theta(b) \pmod{\pi}.$$

We will refer to such a function θ as a π -demonstrator for $\alpha \rightarrow \beta$.

The next lemma is a consequence of the canonical partition theorem of Prömel and Voigt [6]. This lemma was explicitly stated in Example 3.3 of [8] and was used there to prove the instance (see [8, Corollary 5.5]) of the theorem when $L \cong \Pi(A)$.

Lemma 2. Let A be a finite set and let $\gamma: \Pi(A) \rightarrow \Pi(A)$ be the identity representation. Then for each $m \geq 1$ there is n such that $\gamma^n \rightarrow \gamma^m$.

The next lemma is the main ingredient in the proof of the theorem.

Lemma 3. Suppose $\alpha: L \rightarrow \Pi(A)$ is a finite congruence representation. Then there is $n \geq 1$ such that $\alpha^n \rightarrow \alpha$.

Proof. Let $\alpha: L \rightarrow \Pi(A)$ be a finite congruence representation. Let $(A, \mathcal{F})$ be a unary algebra such that $\alpha: L \rightarrow \Pi(A, \mathcal{F})$ is a lattice isomorphism. Without loss of generality, we assume that the identity function is in $\mathcal{F}$ (so $\mathcal{F} \neq \emptyset$) and that $\mathcal{F}$ is closed under composition. Let $m = |\mathcal{F}| \geq 1$. Letting $\gamma: \Pi(A) \rightarrow$

$\Pi(A)$ be the identity function, we can get n by Lemma 2 so that $\gamma^n \rightarrow \gamma^m$. We claim that $\alpha^n \rightarrow \alpha$.

In order to prove that $\alpha^n \rightarrow \alpha$, consider some $\pi \in \Pi(A^n)$. To obtain a π -demonstrator for $\alpha^n \rightarrow \alpha$, let $\mathcal{F} = \{f_0, f_1, \dots, f_{m-1}\}$, and let $F: A \rightarrow A^m$ be the function for which $F(a) = \langle f_0(a), f_1(a), \dots, f_{m-1}(a) \rangle$ for each $a \in A$. Let θ_1 be a π -demonstrator for $\gamma^n \rightarrow \gamma^m$, and then let $\theta = \theta_1 \circ F$. We will show that θ is a π -demonstrator for $\alpha^n \rightarrow \alpha$. Conditions (1) and (2) in the definition need to be proved.

Proof of (1). Suppose $c, d \in A^n$ and $x \in L$. Notice that $\alpha^k = \gamma^k \circ \alpha$ for $1 \leq k < \omega$. Then each of the following equivalences is clear.

$$\begin{aligned} \theta(c) \approx \theta(d) \pmod{\alpha^n(x)} &\Leftrightarrow \theta_1(F(c)) \approx \theta_1(F(d)) \pmod{\gamma^n(\alpha(x))} \\ &\Leftrightarrow F(c) \approx F(d) \pmod{\gamma^m(\alpha(x))} \\ &\Leftrightarrow F(c) \approx F(d) \pmod{\alpha^m(x)} \\ &\Leftrightarrow \text{for each } i < m, f_i(c) \approx f_i(d) \pmod{\alpha(x)} \\ &\Leftrightarrow c \approx d \pmod{\alpha(x)}. \end{aligned}$$

This proves (1) in the definition.

Proof of (2). Suppose that $\pi \in \Pi(A^n)$, and let $\rho \in \Pi(A)$ be the partition such that

$$a \approx b \pmod{\rho} \Leftrightarrow \theta(a) \approx \theta(b) \pmod{\pi}$$

whenever $a, b \in A$. In order to prove (2), we need to show that $\rho \in \Pi(A, \mathcal{F})$. To do this, consider some $i < m$ and arbitrary $a, b \in A$ for which $a \approx b \pmod{\rho}$. Then $\theta(a) \approx \theta(b) \pmod{\pi}$. Hence, there is $\rho_1 \in \Pi(A, \mathcal{F})$ such that $F(a) \approx F(b) \pmod{\gamma^m(\rho_1)}$, so that for each $k < m$, $f_k(a) \approx f_k(b) \pmod{\rho_1}$. Since $\mathcal{F}$ is closed under composition, for each $j < m$, $f_j(f_i(a)) \approx f_j(f_i(b)) \pmod{\rho_1}$. It follows that $F(f_i(a)) \approx F(f_i(b)) \pmod{\gamma^m(\rho_1)}$, so that $\theta_1(F(f_i(a))) \approx \theta_1(F(f_i(b))) \pmod{\gamma^n(\rho_1)}$; consequently, $\theta(f_i(a)) \approx \theta(f_i(b)) \pmod{\pi}$, so that $f_i(a) \approx f_i(b) \pmod{\rho}$. This proves $\rho \in \Pi(A, \mathcal{F})$. $\square$

The following corollary generalizes Lemma 2.

Corollary 4. *Suppose $\alpha: L \rightarrow \Pi(A)$ is a finite congruence representation. Then for each $m \geq 1$ there is n such that $\alpha^n \rightarrow \alpha^m$.*

Proof. This follows immediately from Lemmas 1 and 3 and the observation that $(\alpha^n)^k$ and α^{nk} are isomorphic representations. $\square$

We can now conclude the following corollary, which involves a technical definition from [8]. The reader is referred to [8] for a precise statement of this definition.

Corollary 5. *If L is a finite lattice that has a finite congruence representation, then for each $n < \omega$, L has an n -CPP representation.*

Proof. The n -CPP representation α_n of L is obtained as follows. Let α be a finite congruence representation of L . Let $\alpha_0 = \alpha^2$, which by Lemma 1 is a nontrivial finite congruence representation. Recursively obtain finite congruence representations α_{n+1} so that $\alpha_{n+1} \rightarrow \alpha_n$. This is possible by Lemmas 1 and 3. The representation α_n is an n -CPP representation of L . $\square$

The theorem follows immediately from Corollary 5, Theorem 4.1 of [8], and Remark 5.1 of [8].

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