A SUFFICIENT ARITHMETICAL CONDITION FOR THE IDEAL CLASS GROUP OF AN IMAGINARY QUADRATIC FIELD TO BE CYCLIC

MAKOTO ISHIBASHI

(Communicated by William W. Adams)

ABSTRACT. We give a sufficient arithmetical condition for the ideal class group of an imaginary quadratic field to be cyclic by using Ono's number.

INTRODUCTION

Let $m \leq -5$ be a squarefree integer. Put $K = Q(\sqrt{m})$. Euler observed that the polynomial $x^2 + x + 41$ is a prime for all integer x , $0 \leq x \leq 39$, and Gauss showed that $Q(\sqrt{-163}) = Q(\sqrt{1 - 4(41)})$ has a class number one. Put $f = (1 - m)/4$ if $m \equiv 1 \pmod{4}$ or $= -m$ otherwise, and define $P_K(x) = x^2 + x + f$ if $m \equiv 1 \pmod{4}$ or $= x^2 + f$ otherwise. Then the Frobenius-Ravinovitch theorem says that $P_K(x)$ is prime for $0 \leq x \leq f - 2$ if and only if the class number h_K of K is one.

Sasaki generalized this in the following way. If $n = q_1^{e_1} \cdots q_g^{e_g}$ is the canonical factorization of n into prime powers, write $L(n) = e_1 + \cdots + e_g$ and define

$$L_K = \max L(P_K(n)) : 0 \leq n \leq f - 2.$$

Sasaki [7] proved that $L_K \leq h_K$ and that $L_K = 2$ if and only if $h_K = 2$; the latter result is also implicit in a paper by Carlitz [2]. L_K is known as Ono's number.

The aim of this paper is to prove the following generalization of Sasaki's result:

Theorem. *Let $K = Q(\sqrt{m})$ be an imaginary quadratic field with class number $h_K \geq 3$. Write $h_K = h_1 h_2$, where $h_1 = \prod_{p \parallel h_K} p$ (i.e., the squarefree part of h_K). Suppose that $1 \not\leq h_2$ and put $q_0 = \min\{q; \text{rational prime and } q \mid h_2\}$. If $L_K \geq q_0 + h_K/q_0$, then the ideal class group H_K of K is cyclic.*

1. CZOGALA FILTRATION

For the preliminaries of our proof, let us recall the results of Czogala [3].

Received by the editors October 20, 1990 and, in revised form, March 11, 1991 and June 26, 1991; presented at the Japan Math. Soc. annual meeting, 1986, Kyoto.

1991 *Mathematics Subject Classification.* Primary 11S99.

Definition 1. Let V be the family of all algebraic number fields. Take $F \in V$. For a fixed natural number $n \geq 2$, consider the property

(*)_n if $a_2 a_2 = b_1 \cdots b_k$ for irreducible elements $a_1, a_2, b_1, \dots, b_k$ (for some $k \geq 2$) in O_F (the ring of integers in F), then $k \leq n$.

Write $V_n = \{F \in V; F \text{ has the property } (*)_n\}$.

Lemma 1. Let $(a) = P_1 \cdots P_n$ be the decomposition into prime ideals ($P_1, \dots, P_n$ are not necessarily distinct, $a \in O_F$). Suppose that P_i belongs to an ideal class $x_i \in H_F$ ($1 \leq i \leq n$) and x_0 denotes the principal class of H_F . Then a is an irreducible in O_F if and only if $x_{i_1} \cdots x_{i_m} \neq x_0$ for every proper subset $\{i_1, \dots, i_m\}$ of $\{1, \dots, n\}$.

Lemma 2. Suppose that the principal ideal (a) is expressed by the product of at most n (≥ 2) prime ideals for every irreducible element $a \in O_F$. Then it is true that $F \in V_n$. Conversely, if $F \in V_n$ then the order of each class $x \in H_F$ is at most n .

Lemma 3. If $F \notin V_n$ then $n+1 \leq h_F$.

Proof. By Lemma 2, there exists an irreducible element $a \in O_F$ such that $(a) = P_1 \cdots P_{n+1} A$ for some prime ideals $P_1, \dots, P_{n+1}$ and for some ideal A in O_F . Then it follows that $P_1, P_1 P_2, P_1 P_2 P_3, \dots, P_1, \dots, P_{n+1}$ belong to distinct ideal classes because of Lemma 1. Therefore we have $n+1 \leq h_F$. Q.E.D.

Proposition 1. For every algebraic number field $F \notin V_2$, it is true that $n_F \leq h_F$, where $n_F = \min\{n \geq 2; F \in V_n\} \geq 3$.

Proof. As a consequence of Lemma 3, it follows that

$$V = \bigcup_{n \geq 2} V_n \quad (V_2 \subset V_3 \subset \cdots \subset V_n \subset \cdots).$$

Thus n_F is well defined and $F \notin V_{n_F-1}$. Hence we have $n_F \leq h_F$. Q.E.D.

2. DAVENPORT'S CONSTANT

Let us recall two definitions of Davenport's constant for a finite abelian (additive) group H with order $\#(H)$. Define $u(H)$ as the minimal natural number u such that for any u elements $g_1, \dots, g_u$ (they need not be distinct) in H there exists a nonempty proper subset $\{i_1, \dots, i_m\}$ of $\{1, \dots, u\}$ with vanishing sum (i.e., $g_{i_1} + \cdots + g_{i_m} = 0$). Define $a(H)$ as the maximal length of a sequence $\{g_1, \dots, g_a\}$ of elements (not necessarily distinct) in H such that $g_1 + \cdots + g_a = 0$ that contains no nonempty proper subsequence with vanishing sum. Then one can easily see that $u(H) = a(H) \leq \#(H)$. Especially, for every cyclic group C_n with order n , we have $u(C_n) = a(C_n) = n$.

Lemma 4. Suppose that $H \cong C_{b_1} \oplus \cdots \oplus C_{b_r}$ and $b_i | b_{i+1}$ ($1 \leq i \leq r-1$). Then it is true that $u(H) \leq b_r(1 + \log(h/b_r))$, where $h = \#(H) = b_1 \cdots b_r$. Especially, if $r = 2$, then $u(H) = b_1 + b_2 - 1$.

Proof. See Theorem 7.1 in [1] and Corollary 1.1 in [5]. Q.E.D.

Lemma 5. *If H is a noncyclic finite abelian group then $u(H) \leq \#(H) - 1$.*

Proof. For an arbitrary finite abelian group G and its subgroup K , one can easily prove that $u(G) \leq u(G/K)u(K)$. In our situation, there exists a subgroup K of H such that $K \cong C_p \oplus C_p$ for some prime p . By Lemma 4, we have $u(K) = 2p - 1 \not\leq p^2 = \#(K)$. Thus it follows that $u(H) \leq (2p - 1)u(H/K) \not\leq \#(K) \cdot \#(H/K) = \#(H)$. Q.E.D.

From now on, we assume that F is an imaginary quadratic field $K = Q(\sqrt{m})$ ($m \leq -5$) and $F = K \notin V_2$.

Proposition 2. *For every integer x ($0 \leq x \leq f - 2$), it is true that $x \pm w$, $x \pm w'$ are irreducible elements in K , where w is $(1 + \sqrt{m})/2$ if $m \equiv 1 \pmod{4}$ or $= \sqrt{m}$ otherwise and w' denotes the conjugate element of w .*

Proof. Note that $N_{K/Q}(x + yw) = (x + y/2)^2 - y^2m/4$ if $m \equiv 1 \pmod{4}$ or $= x^2 - y^2m$ otherwise. Hence $y \neq 0$ implies that $N_{K/Q}(x + yw) > -m/4$ if $m \equiv 1 \pmod{4}$ or $> -m$ otherwise. Now assume that $x \pm w = (x_1 + y_1w)(x_2 + y_2w)$ ($y_1 \neq 0$, $y_2 \neq 0$). Then we have $N_{K/Q}(x \pm w) > m^2/16$ if $m \equiv 1 \pmod{4}$ or $> m^2$ otherwise. On the other hand, for every integer x ($0 \leq x \leq f - 2$), we have

$$N_{K/Q}(x \pm w) \leq \begin{cases} f^2 - 3f + 4 = m^2 + 3m + 4 < m^2 \\ \quad (m \leq -5, m \equiv 2, 3 \pmod{4}), \\ f^2 - 2f + 2 = m^2/16 + (6m + 25)/16 < m^2/16 \\ \quad (m < -5, m \equiv 1 \pmod{4}). \end{cases}$$

Thus we have a contradiction. This completes our proof. Q.E.D.

Proposition 3. *The ideal class group H_K of K ($\notin V_2$) is cyclic if and only if $n_K = h_K$.*

Proof. Assume that $H_K = \langle x \rangle$ is cyclic. Generally, it is known that there exists an unramified prime ideal $P \in x$ with degree one (i.e., $N_{K/Q}P = p^1$) and $p \leq \sqrt{|d_K|/3} < f - 1$ (where d_K is the discriminant of K). We may write $P^{h_K} = (a)$, where a is an irreducible element in O_K . Note that p is an irreducible element in O_K . (In fact, if $p = b_1b_2$ is nonunits decomposition, then it follows that $p^2 = N_{K/Q}(p) = N_{K/Q}(b_1)N_{K/Q}(b_2)$. Hence $p = N_{K/Q}(b_1)$, $b_1 = x + wy$ and $p > f - 1$. This is a contradiction.) Since $aa' = N_{K/Q}(a) = p^{h_K}$ and $K \in V_{n_K}$, we have $h_K \leq n_K$. By Proposition 1, it follows that $h_K = n_K$. Conversely, assume that $h_K = n_K$ and that H_K is not cyclic. For every irreducible element $b \in O_K$, we write $(b) = Q_1 \cdots Q_{r_b}$ (the decomposition into prime ideals). By Lemma 5, it follows that $r_b \leq a(H_K) = u(H_K) \leq h_K - 1 = n_K - 1$. Hence we have $K \in V_{n_K-1}$ because of Lemma 2. This contradicts the definition of n_K . Therefore H_K must be cyclic. Q.E.D.

Proposition 4. *It is true that $L_K \leq n_K = a(H_K)$ ($\leq h_K$) for $K \notin V_2$.*

Proof. From Proposition 2 and $P_K(x) = (x + w)(x + w')$, it follows that $L_K \leq n_K$. It is generally true that $K \in V_{a(H_K)}$ because of Lemma 2. Hence we have $n_K \leq a(H_K)$. In the case of imaginary quadratic fields, let $x_1 \cdots x_{a(H_K)} = x_0$ be a maximal length proper relation. Since there exist unramified prime ideals $P_i \in x_i$ ($1 \leq i \leq a(H_K)$) with degree one and $p_i \leq \sqrt{|d_K|/3} < f - 1$, by Lemma

1, it follows that $P_1 \cdots P_{a(H_K)} = (c)$ for some irreducible element $c \in O_K$. Using the same method in the proof of Proposition 3, we know that p_i are irreducible elements in O_K . Hence $a(H_K) \leq n_K$. Thus we have $n_K = a(H_K)$. Q.E.D.

3. PROOF OF THE THEOREM

Let H' be a finite abelian group such that $\#(H') = h_K$ ($= \#(H_K)$). Suppose that $H' \cong C_{b_1} \oplus \cdots \oplus C_{b_r}$, $b_i | b_{i+1}$ ($1 \leq i \leq r-1$), $2 \leq r$, $2 \leq b_1$, $h_K = b_1 \cdots b_r$. Since $q_0 b_1 \leq b_j b_1 \leq h_K$ for some j ($1 \leq j \leq r-1$), it follows that $b_1 - q_0 \leq h_K/q_0 - h_K/b_1$. Hence we have $b_1 + h_K/b_1 - 1 \leq q_0 + h_K/q_0 - 1 = u(C_{q_0} \oplus C_{h_K/q_0})$ because of Lemma 4. Now define a real-valued differentiable function $f(y, z) = yz - \log(yz) - \log y$ with two variables y, z ($2 \leq y, 2 \leq z$). Then $\partial f / \partial z = y - 1/z = 0$, $\partial f / \partial y = z - 2/y = 0$ has no common root. Since $f(y, z) \geq 4 - \log 4 - \log 2 \not\geq 4 - 2.08 \not\geq 1$, we have $b_2 \cdots b_{r-1} - \log(b_2 \cdots b_{r-1}) - \log b_1 \not\geq 1$. Therefore we have

$$\begin{aligned} b_r - b_1 + 1 &\not\geq b_r \not\geq b_r \{b_2 \cdots b_{r-1} - \log(b_2 \cdots b_{r-1}) - \log b_1\} \\ &= h_K/b_1 - b_r \log(h_K/b_r). \end{aligned}$$

This shows that

$$b_r \{1 + \log(h_K/b_r)\} \not\geq b_1 + h_K/b_1 - 1 \leq q_0 + h_K/q_0 - 1.$$

The above argument says that

$$\max\{u(H'); \#(H') = h_K \text{ and } H' \text{ is noncyclic abelian}\} = q_0 + h_K/q_0 - 1.$$

We already know that $L_K \leq n_K = a(H_K) = u(H_K) \leq h_K$. Therefore, $L_K \geq q_0 + h_K/q_0$ implies that $n_K = h_K$. Thus H_K is cyclic because of Proposition 3. Q.E.D.

4. NUMERICAL EXAMPLES

Since our sufficient condition is not available for L_K smaller than $\sqrt{h_K}$, we have only 'limited' applications. However there are some examples for application of our theorem as follows.

In Table 1, the group H_K is cyclic by our theorem. For example, let $m = -199$; then $f = (1-m)/4 = 50$. For the polynomial $P_K(x) = x^2 + x + 50$ ($0 \leq x \leq 50-2$), we have $P_K(21) = 512 = 2^9$. Hence $9 \leq L_K$ ($K = Q(\sqrt{-199})$). On the other hand, it is known that $L_K \leq h_K = 9$. Therefore $L_K = h_K = 9$.

TABLE 1. There are some examples such that $L_K = h_K$ and h_K has a square factor.

$-m$	L_K	h_K
14, 17, 34, 39, 55, 97, 142, 155, 203, 219, 259,	4	4
291, 323, 667, 955, 1027, 1227, 1411, 1507	4	4
183, 295, 583	8	8
199, 367, 1087, 1423	9	9
1255	12	12
3063	16	16

TABLE 2

$-m$	L_K	h_K
419	7	9
491	6	9
563	6	9
1187	6	9

TABLE 3

$-m$	L_K	h_K
479	10	25
599	11	25
1367	13	25
2887	12	25
5503	12	25
7187	10	25
7287	11	25
4591	16	49
7879	15	49

TABLE 4

$-m$	L_K	h_K
362	9	18
519	10	18
527	9	18
679	9	18
1135	11	18
1142	9	18
1207	10	18
1373	9	18
1383	10	18
1687	9	18
1927	10	18
2047	10	18
2167	12	18
3035	10	18
3107	9	18
3543	14	18
7571	9	18
7715	9	18

TABLE 5

$-m$	L_K	h_K
1231	12	27
1399	13	27
3943	13	27
4903	12	27
6007	12	27

In Tables 2 and 3, we have $L_K \geq 3 + 9/3 = 6$, $5 + 25/5 = 10$, $7 + 49/7 = 14$, respectively. Hence H_K is cyclic by our theorem.

In Tables 4 and 5, we have $L_K \geq 3 + 18/3 = 9$, $3 + 27/3 = 12$, respectively. Hence the ideal class group H_K of $K = Q(\sqrt{m})$ is cyclic by our theorem.

Remark. Tables 1–5 were extracted from the unpublished numerical tables on imaginary quadratic number fields $K = Q(\sqrt{m})$ ($-8187 < m < 0$) by H. Wada (1985).

REFERENCES

1. P. C. Baayen, P. Emde Boas, and D. Kruyswijk, *A combinatorial problem of finite abelian groups*. III, ZW 1969-008, Report Math. Centrum, Amsterdam, 1969.
2. L. Carlitz, *A characterization of algebraic number fields with class number two*, Proc. Amer. Math. Soc. **11** (1960), 391–392.

3. A. Czogala, *Arithmetical characterization of algebraic number fields with small class number*, Math. Z. **176** (1981), 247–253.
4. W. Narkiewicz, *Finite abelian groups and factorization problems*, Colloq. Math. **42** (1979), 319–330.
5. J. E. Olson, *A combinatorial problem on finite abelian groups. II*, J. Number Theory **1** (1969), 195–199.
6. G. Rabinovitch, *Eindeutigkeit der zerlegung in primzahl-faktoren in quadratischen zahl-körpern*, Proc. Fifth Internat. Congr. Math., vol. I, 1912, pp. 418–421=J. Reine Angew. Math. **142** (1913), 153–164.
7. R. Sasaki, *On a lower bound for the class number of an imaginary quadratic field*, Proc. Japan Acad. Ser. A Math. Sci. **62** (1986), 37–39.

IWAKI MEISEI UNIVERSITY, 1-27-10 KITAHARA-CHO, TANASHI-SHI, TOKYO 188, JAPAN