POINTWISE MULTIPLIERS OF WEIGHTED BMO SPACES

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Dedicated to Professor Takeshi Kotake on his sixtieth birthday

ABSTRACT. In a recent paper by S. Bloom (*Pointwise multipliers of weighted BMO spaces*, Proc. Amer. Math. Soc. **105** (1989), 950–960), there are some inaccuracies. In this note, we give a counterexample to his “theorem” and a corrected form with proof under a suitable condition on weights. We also give some remarks and examples.

1. INTRODUCTION

Throughout this note, w will be a doubling measure on $\mathbb{R}^n$, i.e., there is a nonnegative function $w(x)$ satisfying the property

$$w(J) = \int_J w(x) dx \leq C w(I),$$

whenever cubes I, J satisfy $I \subset J$ and $|J| \leq 2^n |I|$. Furthermore, we always assume

$$(*) \quad w(x) \geq C_I > 0, \quad x \in I \quad \text{for all cubes } I.$$

(This is equivalent to saying that $f \in L^1_{\text{loc}}(w(x)dx)$ implies $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.) $\mathbb{D}$ will denote the set of all dyadic cubes, i.e.,

$$\mathbb{D} = \left\{ \text{cubes of the form } \prod_{j=1}^n [k_j 2^l, (k_j + 1) 2^l); k \in \mathbb{Z}^n, l \in \mathbb{Z} \right\}.$$

For a cube I and an $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, f_I is defined by $f_I = \frac{1}{|I|} \int_I f(x) dx$. The dyadic maximal operator is

$$M_d^* f(x) = \sup\{|f_I|; x \in I, I \in \mathbb{D}\}.$$

A locally integrable function b is said to belong to BMO_w if

$$\|b\|_{\text{BMO}_w} = \sup \left\{ \frac{1}{w(I)} \int_I |b(x) - b_I| dx \right\} < +\infty,$$

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where the supremum is taken over all cubes I with sides parallel to the coordinate axes.

A locally integrable function b is said to belong to $\text{BMO}_{w,d}$ if

$$\|b\|_{\text{BMO}_{w,d}} = \sup_{I \in \mathbb{D}} \left\{ \frac{1}{w(I)} \int_I |b(x) - b_I| dx \right\} < +\infty.$$

A w -atom is a function supported in a cube I with $\int a(x) dx = 0$ and $\|a\|_\infty \leq 1/w(I)$. $f \in H_w^1$ if there exist w -atoms $\{a_j\}$ and a sequence $\{\lambda_j\} \in l^1$ such that $f = \sum \lambda_j a_j$ in $L^1(w)$, and we set

$$\|a\|_{H_w^1} = \inf \left\{ \sum |\lambda_j|; f = \sum \lambda_j a_j \right\}.$$

Similarly, a (w, d) -atom is a function supported in a cube $I \in \mathbb{D}$ with $\int a(x) dx = 0$ and $\|a\|_\infty \leq 1/w(I)$. $f \in H_{w,d}^1$ if there exist (w, d) -atoms $\{a_j\}$ and a sequence $\{\lambda_j\} \in l^1$ such that $f = \sum \lambda_j a_j$ in $L^1(w)$, and we set

$$\|a\|_{H_{w,d}^1} = \inf \left\{ \sum |\lambda_j|; f = \sum \lambda_j a_j, a_j : (w, d)\text{-atoms} \right\}.$$

For our weight w , we have the following (cf. [6]).

Proposition 1. $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ belongs to $H_{w,d}^1$ if and only if $M_d^* f \in L^1(w)$, and $\|M_d^* f\|_{L^1(w)}$ is equivalent to $\|f\|_{H_{w,d}^1}$.

We note that this proposition does not hold for general doubling measures, even if $w \in A_p(\mathbb{R}^n)$ ($1 < p$ and $pn < n + 1$) (hence $w \in D_{pn}$ in the sense of Strömberg-Torchinsky [5]) and the convergence of atoms in the definition of H_w^1 is taken in the sense of distributions, see Example 1 in §3. Here A_p is Muckenhoupt's weight class.

Next, modifying the proof of the H^1 -BMO duality by Coifman and Weiss [2, pp. 631–634], one can show

Proposition 2. Every $b \in \text{BMO}_{w,d}$ determines a bounded linear functional on $H_{w,d}^1$ and the dual form is given by

$$\langle f, b \rangle = \sum \lambda_j \int a_j(x) b(x) dx,$$

where $\sum \lambda_j a_j$ is any atomic decomposition of f . Furthermore

$$\langle f, b \rangle = \int f(x) b(x) dx$$

in the case where f is a finite linear combination of (w, d) -atoms and $b \in \text{BMO}_{w,d}$, and in the case where $f \in H_{w,d}^1$ and $b/w \in L^\infty(\mathbb{R}^n)$.

Now, as for pointwise multipliers of $H_{w,d}^1$ we have

Theorem 1. $\phi(x)f(x) \in H_{w,d}^1$ for every $f \in H_{w,d}^1$ if and only if

$$(1) \quad \phi \in L^\infty$$

and

$$(2) \quad \int_I |\phi(x) - \phi_I| dx \leq C w(I) \left/ \int_{I^c} \frac{w(x) dx}{|x - x_I|^n} \right. \text{ for all } I \in \mathbb{D},$$

where x_I is the center of the cube I .

The proof of Theorem 1.4 in Bloom [1] is valid in this case, in virtue of Proposition 2. His statement for BMO_w in his Theorem 1.7, however, is not correct; a counterexample will be given later. His Theorem 1.4 should read as follows.

Theorem 2. Suppose $\int \frac{w(x)dx}{1+|x|^n} < +\infty$. Then $\phi(x)b(x) \in \text{BMO}_{w,d}$ for every $b \in \text{BMO}_{w,d}$ if and only if

$$(1) \quad \phi \in L^\infty$$

and

$$(2') \quad \int_I |\phi(x) - \phi_I| dx \leq C w(I) / \left(1 + \int_{I^c} \frac{w(x) dx}{|x - x_I|^n} \right) \quad \text{for all } I \in \mathbb{D}.$$

Remark. (1) + (2') is equivalent to (1') $\phi \in L^\infty \cap \text{BMO}_{w,d}$ + (2).

2. PROOF OF THEOREM 2

For the proof we prepare several lemmas. Since $M_d^* f \leq C(1 + |x|^n)^{-1}$ for every bounded function f with compact support, we have by Proposition 1

Lemma 1. Suppose $\int \frac{w(x)dx}{1+|x|^n} < +\infty$. Then every bounded function with compact support belongs to $H_{w,d}^1$.

Lemma 2. Let $w(x)$ be as in Lemma 1. Then if $b \in \text{BMO}_{w,d}$ we have for any cube $I \in \mathbb{D}$

$$\lim_{j \rightarrow \infty} \frac{1}{|I^j|} \int_{I^j} b(x) dx = b_{\infty,I} \quad \text{and} \quad \lim_{j \rightarrow \infty} \frac{1}{|I^j|} \langle \chi_{I^j}, b(x) \rangle = b_{\infty,\infty,I}$$

exist, where $I^1 = I$ and I^j is the unique cube $J \in \mathbb{D}$ with $I \subset J$ and $|J| = 2^n |I^{j-1}|$, $j = 2, 3, \dots$

Proof (by A. Uchiyama). We may assume I is of the form $[0, 2^m)^n$. Let $I \subset J \in \mathbb{D}$. Then

$$\begin{aligned} \left| \frac{1}{|J|} \int_J b dy - \frac{1}{|J^2|} \int_{J^2} b dy \right| &= 2^n \cdot \frac{1}{|J^2|} \int_J |b - b_{J^2}| dy \\ &\leq C \frac{1}{|J^2|} \int_{J^2} |b - b_{J^2}| dy \leq C \|b\|_{\text{BMO}_{w,d}} \frac{w(J^2)}{|J^2|} \\ &\leq C \|b\|_{\text{BMO}_{w,d}} \int_{J^2 \setminus J} \frac{w(x)}{|x|^n} dx. \end{aligned}$$

Hence

$$\sum_{j=2}^{\infty} \left| \frac{1}{|I^j|} \int_{I^j} b dy - \frac{1}{|I^{j-1}|} \int_{I^{j-1}} b dy \right| \leq C \|b\|_{\text{BMO}_{w,d}} \int_{\mathbb{R}^n} \frac{w(x)}{(1+|x|)^n} dx < +\infty.$$

Thus, $b_{\infty,I}$ exists.

Next, since $\int (\frac{1}{|J|} \chi_J - \frac{1}{|J^2|} \chi_{J^2}) dx = 0$, $\text{supp } \frac{1}{|J|} \chi_J - \frac{1}{|J^2|} \chi_{J^2} \subset J^2$, we see that $|J|(\frac{1}{|J|} \chi_J - \frac{1}{|J^2|} \chi_{J^2})/w(J^2)$ is a (w, d) -atom, and so by Proposition 2

$$\left| \left\langle \frac{1}{|J|} \chi_J, b \right\rangle - \left\langle \frac{1}{|J^2|} \chi_{J^2}, b \right\rangle \right| \leq C \frac{w(J^2)}{|J|} \|b\|_{\text{BMO}_{w,d}}.$$

Thus, as before, we see that $b_{\infty,\infty,I}$ exists. Q.E.D.

Now there are 2^n quadrants $K(j)$ with respect to $\mathbb{D}$, such that each $I \in \mathbb{D}$ is contained uniquely in one of them. For $b \in \text{BMO}_{w,d}$ let

$$b_{\infty, K(j)} = b_{\infty, I} \quad \text{and} \quad b_{\infty\infty, K(j)} = b_{\infty\infty, I},$$

where $I \subset K(j)$. Note that these are independent of the choice of $I \subset K(j)$. Set

$$b_{\infty} = \sum_{j=1}^{2^n} b_{\infty, K(j)} \chi_{K(j)} \quad \text{and} \quad b_{\infty\infty} = \sum_{j=1}^{2^n} b_{\infty\infty, K(j)} \chi_{K(j)}.$$

The following lemma is proved easily.

Lemma 3. *Let w be as in Lemma 1. Suppose $b_j \in \text{BMO}_{w,d}$ and $b_j \rightarrow b$ in $\text{BMO}_{w,d}$. Then*

$$\lim_{j \rightarrow \infty} (b_j)_{\infty\infty} = b_{\infty\infty},$$

and there exists a subsequence $\{j_l\}$ such that for almost all $y \in \mathbb{R}^n$

$$\lim_{l \rightarrow \infty} (b_{j_l}(y) - (b_{j_l})_{\infty}) = b(y) - b_{\infty}.$$

Using this and the fact that $b - b_{\infty} + b_{\infty\infty} = 0$ if b is constant on each quadrant, we have by the closed graph theorem

Lemma 4. *Let w be as in Lemma 1. Suppose $\phi(b - b_{\infty} + b_{\infty\infty}) \in \text{BMO}_{w,d}$ for every $b \in \text{BMO}_{w,d}$. Then*

$$\|\phi(b - b_{\infty} + b_{\infty\infty})\|_{\text{BMO}_{w,d}} \leq C \|b\|_{\text{BMO}_{w,d}}$$

holds.

Using Lemma 1 and Proposition 2, we have

Lemma 5. *Let w be as in Lemma 1. Suppose $\phi f \in H_{w,d}^1$ for every $f \in H_{w,d}^1$. Then for any $I \in \mathbb{D}$*

$$\langle \phi f, b \rangle = \int f \phi \left(b - b_I + \left\langle \frac{1}{|I|} \chi_I, b \right\rangle \right) dx$$

for all $b \in \text{BMO}_{w,d}$ and all $f \in H_{w,d}^1$, whose atoms are of finite number and are contained in I .

Using Lemma 3, Lemma 5, and Proposition 2, we have

Lemma 6. *Let w and ϕ be as in Lemma 5. Then, if $b \in \text{BMO}_{w,d}$, it follows that*

$$\begin{aligned} \phi(b - b_{\infty} + b_{\infty\infty}) &\in \text{BMO}_{w,d}, \\ \|\phi(b - b_{\infty} + b_{\infty\infty})\|_{\text{BMO}_{w,d}} &\leq \|M_{\phi}\|_{H_{w,d}^1, H_{w,d}^1} \|b\|_{\text{BMO}_{w,d}}, \end{aligned}$$

where M_{ϕ} is the multiplication operator by ϕ .

Now using Proposition 2, Lemma 4, and Lemma 1, one can show

Lemma 7. *Let w be as in Lemma 1. Then, if there exist $\phi \in L^\infty(\mathbb{R}^n)$ and $\phi(b - b_\infty + b_{\infty\infty}) \in \text{BMO}_{w,d}$ for every $b \in \text{BMO}_{w,d}$, it follows that $\phi f \in H_{w,d}^1$ for every $f \in H_{w,d}^1$.*

The following lemma can be proved in the same way as in the proof of Theorem 1(1), using Proposition 2 and Lemma 1.

Lemma 8. *Let $w(x)dx$ be a doubling measure satisfying $(*)$. Then, if $\phi b \in \text{BMO}_{w,d}$ for every $b \in \text{BMO}_{w,d}$, it follows that $\phi \in L^\infty(\mathbb{R}^n)$.*

Now we are in a position to prove Theorem 2.

The proof of Theorem 2. Suppose $\phi b \in \text{BMO}_{w,d}$ for every $b \in \text{BMO}_{w,d}$. Then, since $1 \in \text{BMO}_{w,d}$, we have $\phi \in \text{BMO}_{w,d}$. Next, using Lemmas 8 and 7, we see that $\phi f \in H_{w,d}^1$ for every $f \in H_{w,d}^1$. Hence, by Theorem 1, ϕ satisfies (1) and (2). Thus ϕ satisfies (1) and (2').

Next, suppose ϕ satisfies (1) and (2'), and let $b \in \text{BMO}_{w,d}$. Then $\phi \in \text{BMO}_{w,d}$, and by Theorem 1, $\phi f \in H_{w,d}^1$ for every $f \in H_{w,d}^1$. Thus by Lemma 6 $\phi(b - b_\infty + b_{\infty\infty}) \in \text{BMO}_{w,d}$. Since b_∞ and $b_{\infty\infty}$ are constant on each quadrant and $\phi \in \text{BMO}_{w,d}$, it follows that $\phi b \in \text{BMO}_{w,d}$. Q.E.D.

As an easy consequence we have

Proposition 3. *Let $w(x)$ be as in Theorem 2. Then, if $\phi \in L^\infty(\mathbb{R}^n)$ satisfies (2') for all cubes I , it follows that $\phi(x)b(x) \in \text{BMO}_w$ for every $b \in \text{BMO}_w$.*

Proof. For $t \in \mathbb{R}^n$, let $\mathbb{D}_t = \{I = J + t; J \in \mathbb{D}\}$ and introduce $H_{w,d,t}^1$ and $\text{BMO}_{w,d,t}$ by replacing $\mathbb{D}$ by $\mathbb{D}_t$. Then, Propositions 1 and 2, Theorems 1 and 2, and Lemmas 1 through 8 hold. Furthermore, the equivalence in Proposition 1 is independent of $t \in \mathbb{R}^n$, and if ϕ satisfies (1) and (2'), the operator norm $\|M_\phi\|_{H_{w,d,t}^1, H_{w,d,t}^1}$ is bounded by a constant independent of $t \in \mathbb{R}^n$. Now, let I be a cube. Then, there exist $t \in \mathbb{R}^n$ and $J, K \in \mathbb{D}_t$ such that $J \subset I \subset J^2$, $|K| = 1$, $\text{dist}(0, K) = 0$, and $K \subset J^k$ for some $k \in \mathbb{N}$. We have by Lemma 6

$$\begin{aligned} \frac{1}{w(I)} \int_I |\phi b - (\phi b)_I| dx &\leq \frac{C}{w(J^2)} \int_{J^2} |\phi b - (\phi b)_{J^2}| dx \\ &\leq C \|M_\phi\|_{H_{w,d,t}^1, H_{w,d,t}^1} \|b\|_{\text{BMO}_{w,d,t}} + C(|b_{\infty, J^2}| + |b_{\infty\infty, J^2}|) \|\phi\|_{\text{BMO}_{w,d,t}}. \end{aligned}$$

Note that $b_{\infty, J^2} = b_{\infty, K}$ and $b_{\infty\infty, J^2} = b_{\infty\infty, K}$. Hence, as in the proof of Lemma 2 we have

$$\begin{aligned} |b_{\infty, J^2}| &\leq C \|b\|_{\text{BMO}_{w,d,t}} \int_{K^c} \frac{w(x)}{|x - x_K|^n} dx + |b_K| \\ &\leq C \|b\|_{\text{BMO}_w} \int \frac{w(x)}{(1 + |x|)^n} dx + C \int_{|x| \leq 1} |b(x)| dx. \end{aligned}$$

Since $\|\chi_K\|_{H_{w,d,t}^1} \leq C \|M_{d,t}^* \chi_K\|_{L^1(w)} \leq C \int \frac{w(x) dx}{(1 + |x|)^n}$, we have as above

$$|b_{\infty\infty, J^2}| \leq C \|b\|_{\text{BMO}_{w,d,t}} \int \frac{w(x)}{(1 + |x|)^n} dx.$$

Thus we have

$$\frac{1}{w(I)} \int_I |\phi b - (\phi b)_I| dx \leq C \left(\|b\|_{\text{BMO}_w} + \int_{|x| \leq 1} |b(x)| dx \right). \quad \text{Q.E.D.}$$

3. REMARKS

A counterexample to Theorem 1.7 of Bloom [1]. For the sake of simplicity, we consider the one-dimensional case. Let $w(x) = 1/(1 + |x|^{1/2})$ and $\phi(x) \in C^\infty(\mathbb{R})$ such that $\phi(x - \frac{1}{2})$ is odd, $\phi(x) = 1$ ($x \geq 1$), and $\|\phi\|_\infty \leq 1$. Then one can show that ϕ satisfies (1) and (2) in Theorem 1, but $\phi \notin \text{BMO}_w$. In fact, for $a > 2$

$$\int_{1-a}^a |\phi(x) - \phi_{(1-a, a)}| dx \geq 2a - 2, \quad - \int_{1-a}^a \frac{1}{1 + |x|^{1/2}} dx \leq C a^{1/2}.$$

This implies $\phi \notin \text{BMO}_w$. Next, one sees easily

$$\int_{I^c} \frac{w(x)}{|x - x_I|} dx \leq C |I|^{-1/3} \quad \text{for } I : \text{intervals.}$$

We also have

- (i) If $I \cap [0, 1] = \emptyset$, $\int_I |\phi - \phi_I| dx = 0$.
- (ii) If $I \cap [0, 1] \neq \emptyset$ and $|I| < 4$, then

$$\int_I |\phi - \phi_I| dx \leq C \|\phi\|_{\text{Lip } 1} |I|^2.$$

Also $w(I) \sim |I|$. Hence

$$\frac{w(I)}{\int_{I^c} \frac{w(x)}{|x - x_I|} dx} \geq C |I|^{1+1/3} \geq C |I|^2 \geq C \int_I |\phi - \phi_I| dx.$$

- (iii) If $I \cap [0, 1] \neq \emptyset$ and $|I| \geq 4$, then

$$\begin{aligned} w(I) &= \int_I \frac{dx}{1 + |x|^{1/2}} \leq 2 + \int_1^{|I|} \frac{dt}{\sqrt{t}} \leq 2 |I|^{1/2}, \\ w(I) &\geq \int_1^{|I|} \frac{dx}{1 + |x|^{1/2}} \geq \frac{1}{2} \int_1^{|I|} \frac{dt}{\sqrt{t}} \geq |I|^{1/2} - 1 \geq \frac{|I|^{1/2}}{2}, \\ \int_{I^c} \frac{w(x)}{|x - x_I|} dx &= \int_{(2I)^c} \frac{w(x)}{|x - x_I|} dx + \int_{2I \setminus I} \frac{w(x)}{|x - x_I|} dx \\ &\leq C |I|^{-1/2} + C |I|^{-1} w(2I) \leq C |I|^{-1/2}. \end{aligned}$$

Hence

$$\frac{w(I)}{\int_{I^c} \frac{w(x)}{|x - x_I|} dx} \geq C |I| \geq C \int_I |\phi(x) - \phi_I| dx.$$

Therefore, ϕ satisfies conditions (1) and (2) for $w(x) = 1/(1 + |x|^{1/2})$.

Remark 1. The assumption (*) is used to prove Propositions 1 and 2. The equivalence of $\|f\|_{H_{w,d}^1}$ and $\|M_d^* f\|_{L^1(w)}$ does not hold for general doubling weights. We give an example.

Example 1. For $w(x) = |x - 1|$, $w(x) \in A_\infty(\mathbb{R})$ and hence $w(x)$ is a doubling measure. For this $w(x)$ there exists no $C > 0$ such that

$$\|M_d^* a\|_{L^1(w)} \leq C \|a\|_{H_{w,d}^1} \quad \text{for all } (w, d)\text{-atoms } a.$$

In fact, for $j \in \mathbb{N}$ let

$$\begin{aligned} a_j(x) &= 2^j (\chi_{[1-2^{-j+1}, 1-2^{-j})} - \chi_{[1+2^{-j}, 1+2^{-j+1})}), \\ \varphi_j^+(x) &= 2^{j+1} \chi_{[1+2^{-j-1}, 1+2^{-j})} - 2^j \chi_{[1+2^{-j}, 1+2^{-j+1})}, \\ \varphi_j^-(x) &= -2^{j+1} \chi_{[1-2^{-j}, 1-2^{-j-1})} + 2^j \chi_{[1-2^{-j+1}, 1-2^{-j})}. \end{aligned}$$

Then $a_j^+(x) = \frac{\varphi_j^+(x)}{2^{j+1}w([1, 1+2^{-j+1}))}$ and $a_j^-(x) = \frac{\varphi_j^-(x)}{2^{j+1}w([1-2^{-j+1}, 1])}$ are (w, d) -atoms. Let $\lambda_j = 2^{j+1}w([1, 1+2^{-j+1})) = 2^{j+1}w([1-2^{-j+1}, 1])$. Then

$$\sum_{k=j}^{\infty} \lambda_k \leq \sum_{k=j}^{\infty} 2^{k+1} \int_0^{2^{-k+1}} t \, dt = \frac{2^{2-j}}{(1-2^{-1})},$$

$$a_j(x) = \sum_{k=j}^{\infty} \lambda_k a_k^+(x) + \sum_{k=j}^{\infty} \lambda_k a_k^-(x) \quad \text{in } \mathcal{S}'(\mathbb{R}) \text{ (and in } L^1(w)).$$

Hence $\|a_j\|_{H_{w,d}^1} \leq C 2^{-j}$. On the other hand, for $x \in [0, 1)$ we have $M_d^* a_j(x) \geq \left| \int_0^1 a_j(t) \, dt \right| = 1$, and so $\int_{-\infty}^{\infty} M_d^* a_j(x) w(x) \, dx \geq \int_0^1 w(x) \, dx = \frac{1}{2}$. Since $2^{-j+1} a_j(x)/w([0, 2))$ is a (w, d) -atom, these show our assertion.

Remark 2. E. Nakai characterizes the pointwise multipliers of BMO_w in the case $w(I(x, sr)) \leq C s^{n+1} w(I(x, r))$, $s \geq 1$, $x \in \mathbb{R}^n$, $r > 0$ (where $I(x, r)$ is the cube with center x and sidelength r , this condition is called D_{n+1} in [5]), including the case $\int \frac{w(x) \, dx}{1+|x|^n} = +\infty$. He also considers more general BMO spaces than ours. By our method, the converse of Proposition 3 can be derived under the condition $w \in D_{n+1}$, and hence is included in the work of Nakai.

However, there are doubling weights that satisfy $(*)$ and $\int \frac{w(x) \, dx}{1+|x|^n} < \infty$, but does not satisfy D_{n+1} . We give an example in the one-dimensional case, for the sake of simplicity, however, the proof we omit.

Example 2. Let $1 < r < +\infty$ and $0 < \alpha < \beta < 1$. Set

$$w(x) = \begin{cases} \min(|x|^{-\alpha}, (2^{-j\beta/(r-1)} + |x - 2^j|)^{r-1}), & |x - 2^j| < 1, \\ |x|^{-\alpha}, & \text{otherwise.} \end{cases} \quad j = 2, 3, \dots,$$

Then, $w \in A_p$ ($p > r$) but $w \notin A_r$, $w \notin D_s$ ($s < r$). It also holds that $w(x) \geq |x|^{-\beta}$ and $\int \frac{w(x) \, dx}{1+|x|} < +\infty$.

Remark 3. Multiplication operators are not well defined on cosets of BMO . So, BMO should be a set of functions, when one says “ ϕ multiplies BMO ”, etc. In this context, even in the unweighted case the set of pointwise multipliers of $\text{BMO}(\mathbb{R}^n)$ contains nonconstant functions (cf. [4]), which contradicts Bloom’s Theorem 1.6. Although we cannot trust his theorems concerning BMO_w in [1], his ideas are good ones. Our work depends heavily on his paper.

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REFERENCES

1. S. Bloom, *Pointwise multipliers of weighted BMO spaces*, Proc. Amer. Math. Soc. **105** (1989), 950–960.
2. R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645.
3. E. Nakai, *Pointwise multipliers for functions of weighted bounded mean oscillation*, Studia Math. (to appear).
4. E. Nakai and K. Yabuta, *Pointwise multipliers for functions of bounded mean oscillation*, J. Math. Soc. Japan **37** (1985), 207–218.
5. J.-O. Strömberg and A. Torchinsky, *Weighted Hardy spaces*, Lecture Notes in Math., vol. 1381, Springer-Verlag, Berlin and Heidelberg, 1989.
6. A. Uchiyama, *Lecture notes (285I)*, Department of Mathematics, University of California, Los Angeles, 1980–1981.

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